Introduction to PreCalculus

Ques. Why study PreCalculus?

Ans. Because it is a prerequisite for Calculus.

Ques. Why study Calculus?

There are several standard answers:

It is an introduction to higher mathematics.

It studies the infinitely large and the infinitely small which lies outside the scope of elementary mathematics.

It is used to explain the elliptical orbits of planets, motion of bodies, gravity etc.

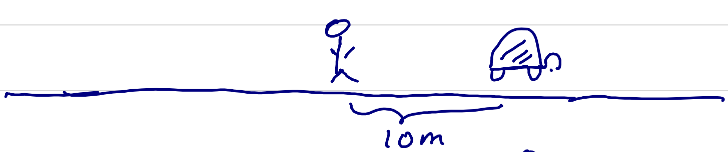
It is an essential tool in Physics, Chemistry, Biology Statistics, Computer Science etc.

The list goes on but I want to give to you some glimpses of ideas from Calculus to instill some motivation.

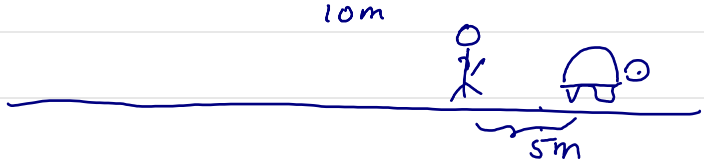
(I) Zeno's paradox

Achilles is about to race a tortoise. He gives the tortoise a 100m head start.

After a while



After a while

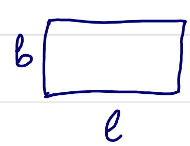


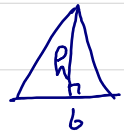
After some time Achilles reaches the starting point of the tortoise. Meanwhile the tortoise has moved a certain distance, say 10m. Then Achilles runs towards the tortoise and reaches the 110 m mark. Meanwhile the tortoise has moved a certain distance say 5 m. This process continues.

<u>Ques.</u> Can Achilles even reach the tortoise, despite his agility?

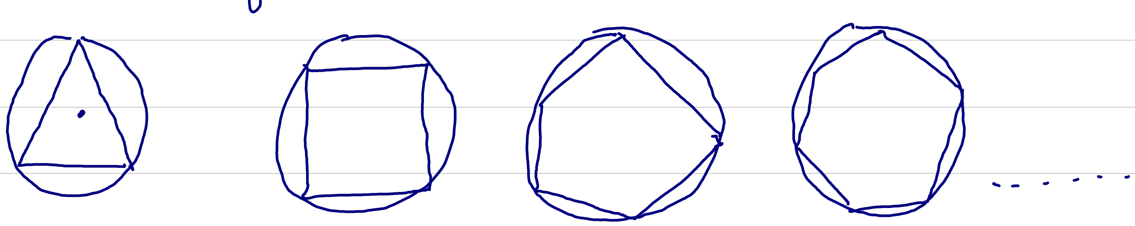
(II) <u>Area of Circle</u>

We were told in school that the area of a circle with radius R is equal to πR^2.
It was Archimedes who derived it. It is easy to convince ourselves that

Area () = $l \cdot b$

Area () = $\frac{1}{2} \cdot b \cdot h$.

Archimedes came up with the following idea: Inscribe a regular polygon inside the circle and increase the number of sides.



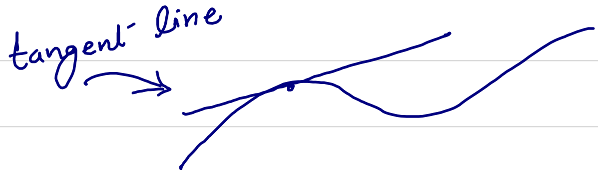
$n=3$ $n=4$ $n=5$ $n=6$ . . . . . . $n \to \infty$

Notice that as you increase the number of sides the inscribed polygon has an area closer to the circle. For very large n, the area of the polygon is very close to the area of the circle.

Archimedes knew how to find the area of each polygon by cutting them up into triangles. In this way he got an approximation for the number π and hence found the formula for area of a circle.

(III) Tangents

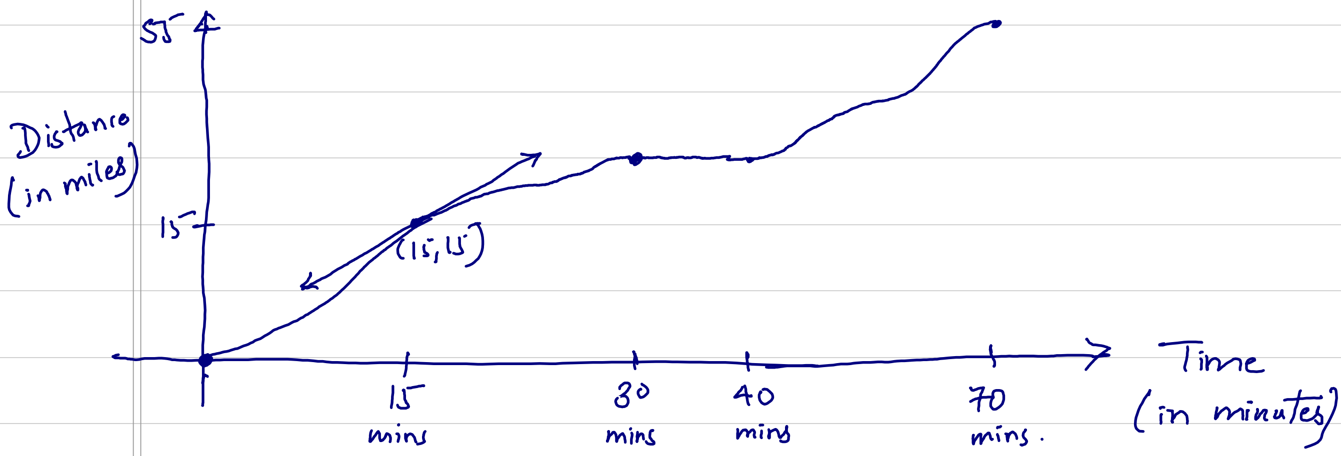
A tangent to a curve at a given point P is the line that near P intersects the curve only at P.



Calculus studies functions and the tangents. But why are tangents important?

Because they give us the instantaneous rate of change, for instance, the velocity at a certain time.

Say I make a road trip from LAF to BTR and I record the distance I travelled with respect to time. Say I graph the Distance Vs Time plot:



Ques. What was my velocity 15 mins. after I left LFT?

Ans. It is given by the slope of the tangent at the point (15,15).

This is why tangents are very important.

Functions

Functions are the bread and butter of Calculus. So our goal in MATH 109 we will learn about quadratic functions, polynomial functions, rational functions exponential and logarithmic functions.

In MATH 110 you will learn about trigonometric functions.

We learnt in MATH 105 that function is a correspondence between two variables, usually called x and y. Recall that variable is a quantity that can assume different values.

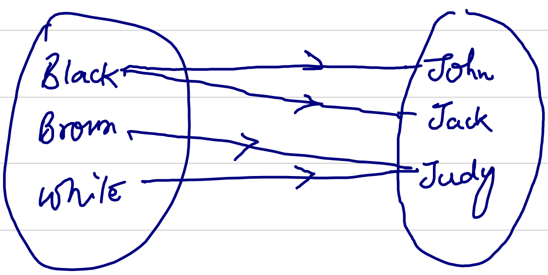
x is called the independent variable / input.

y is called the dependent variable / output.

Precise definition:

Def. A function from X to Y is a rule / relation that associates each element of X with a $\boxed{\text{some element}}$ of Y. Here X and Y are sets. A set is just a collection of objects.

There is a subtle problem with the above definition. I will illustrate the problem with an example. Say I take X to be the set {Black, Brown, White} and Y to be the set {John, Jack, Judy}. And the correspondence between X and Y is the color of their hair. Say it looks like this



Ques. Does this example satisfy the above mentioned definition?

Ans. Yes.

Ques. What is wrong?

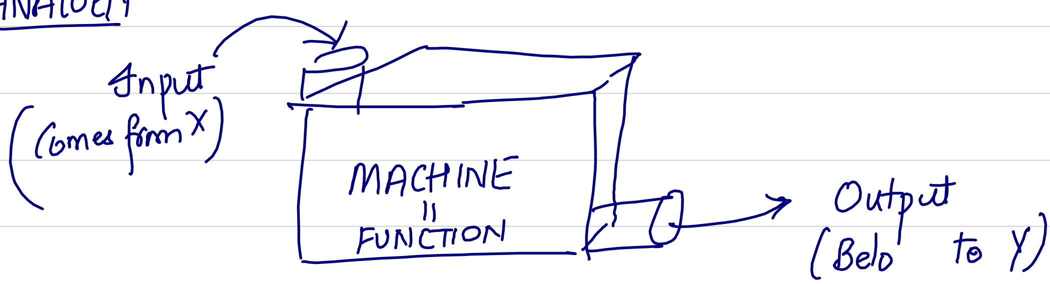
EXERCISE 1.

[Note: Sets are represented as ovals and arrows denote the correspondence.]

So the problem lies in how Black maps to two people: John and Jack. This kind of correspondence is not allowed in functions. So as a remedy, we have to replace the phrase [some element] by the phrase: [a unique element]

The objects that satisfy the previous definition are called <u>relations</u>. So the previous example is a relation but not a function.

<u>ANALOGY</u>

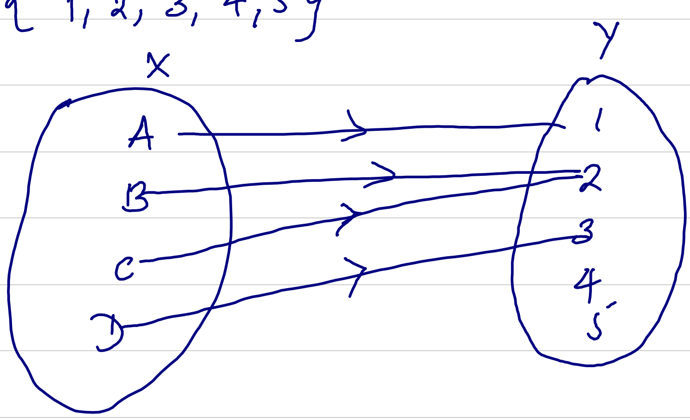


<u>EXERCISE 2</u>

Is the following a function?

$X = \{A, B, C, D\}$

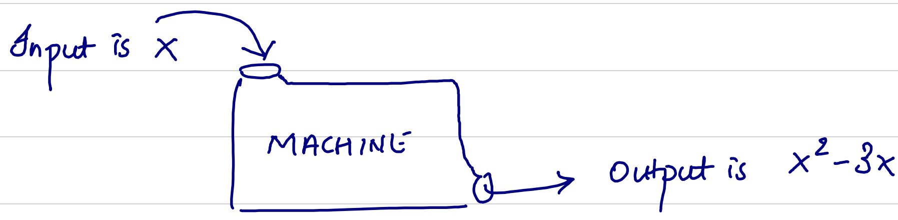
$Y = \{1, 2, 3, 4, 5\}$



In Calculus, we will encounter mostly numerical functions, i.e. functions that take numbers as input and output numbers.

Ex: $f(x) = x^2 - 3x$

Using the analogy of the machine we have:



So if I input x, I get $x^2 - 3x$

You can think of the independent variable or argument as a placeholder. So for example, $f(x) = x^2 - 3x$ can be thought of as

$$f(\square) = (\square)^2 - 3(\square).$$

So whatever you input into this function, it will square it and subtract three times that number from its square.

Hence,

$$f(1) = 1^2 - 3\cdot 1 = 1 - 3 = -2$$

$$f(2) = 2^2 - 3\cdot 2 = 4 - 6 = -2$$

$$f(x+1) = (x+1)^2 - 3(x+1)$$

$$f(-x) = (-x)^2 - 3(-x) = x^2 + 3x$$

Note:

$f(x)$ is read as "f evaluated at x" or "f of x".

It represents the y-value that corresponds to a particular x-value.

EXERCISE 3

Let $h(x) = x^2 + 2x$.

(a) $h(x+1) = ?$

(b) $h(x) + h(1) = ?$

This exercise shows that $h(x+y) \neq h(x) + h(y)$ in general.

EXERCISE 4

Let $g(t) = t^2 - t$

(a) $g(-t) = ?$

(b) $-g(t) = ?$

This exercise shows that $g(-t) \neq -g(t)$ in general.

EXERCISE 5

Let $f(x) = 3x+5$

(a) $f\left(\frac{1}{2}\right) = ?$

(b) $\dfrac{f(1)}{f(2)} = ?$

This exercise shows that $f\left(\frac{x}{y}\right) \neq \dfrac{f(x)}{f(y)}$ in general.

Domain, Codomain and Range

Consider the function $A(r) = \pi r^2$, where $A(r)$ gives the area of the circle with radius r.

Ques.: what are the possible values of r?

Ans: r is a variable. But surely the radius cannot be a negative number. If $r = 0$ then we get a degenerate circle, i.e. a point. All real numbers greater than 0 are viable numbers. Thus, the possible values are all real numbers greater than or equal to 0.

Def: Let f be a function from a set X to a set Y. The set X is called the domain of f. In other words, the domain of a function is the set of input values.

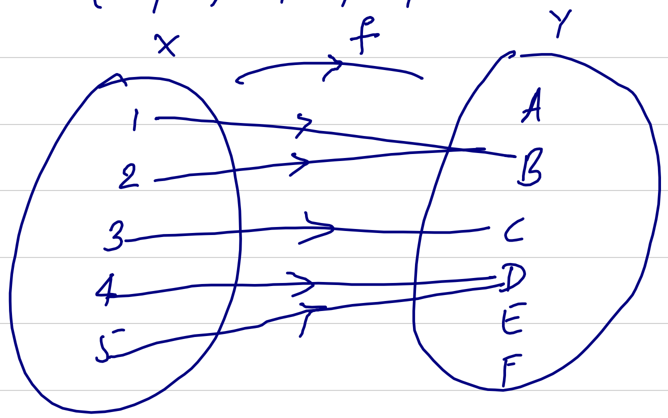
Def: Let f be the same as in previous definition. The set Y is called the codomain of f.

Def: Let f be defined as above. The range of f is the set of all elements y in Y that have atleast one element x such that $f(x) = y$. In other words it is the set of output values.

Ex:

$X = \{1, 2, 3, 4, 5\}$

$Y = \{A, B, C, D, E, F\}$



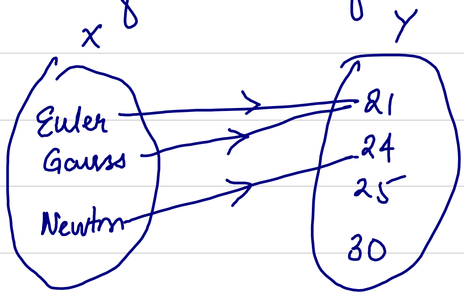
Domain = $\{1, 2, 3, 4, 5\}$

Codomain = $\{A, B, C, D, E, F\}$

Range = $\{B, C, D\}$

EXERCISE 6 Let $X = \{$ Euler, Gauss, Newton $\}$, $Y = \{21, 24, 25, 30\}$.

Let f be a function defined as follows:



Find the Domain, Codomain and Range of f.

Let's go back to the area example. We have

$$A(r) = \pi r^2.$$

Ques. What is the range of the function A?

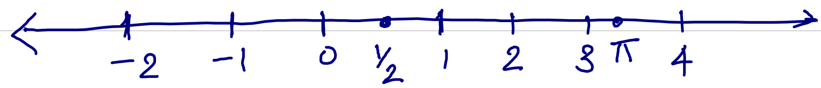
EXERCISE 7

Note the we have complete freedom in specifying a set as the codomain for a function if the codomain is not mentioned specifically. But note that it must include the entire range. For instance, in the above example, the codomain of A can be any set of the real line that contains the range. I could even take the codomain to be $\mathbb{R}$. Henceforth, we will be interested only in the domain and range of a function.

Before we proceed to find domains of some numerical functions, we must learn the interval notation and the union symbol ($\cup$). Previously in MATH 105 you described sets of the real line in words or using phrases like all x such that $1 \leq x \leq 4$. But interval notation is more efficient than this.

Interval notation and the union symbol $\cup$

Real line $\mathbb{R}$



You have already encountered the real line. It is the set consisting of all real numbers. Later on we will encounter complex numbers which don't lie on the line. Note that unlike a slab of a metal or some object which has pores or empty space between their atoms, the real line does not have any gaps. Note the irony in the name.

Set notation of intervals:

$$[1,2] = \{x \in \mathbb{R} : 1 \le x \le 2\}.$$

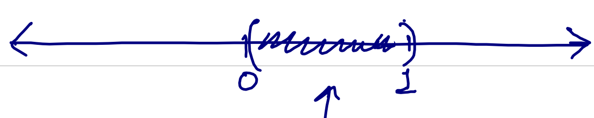
This is read as all $x \in \mathbb{R}$ such that $1 \le x \le 2$.

If you want to write $[1,2]$ descriptively, you would have

$[1,2] = \{$ the set of all real numbers x such that $x \ge 1$ and $x \le 2\}$.

$$(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$$

Note that 0 and 1 are not in this set.



x lies here.

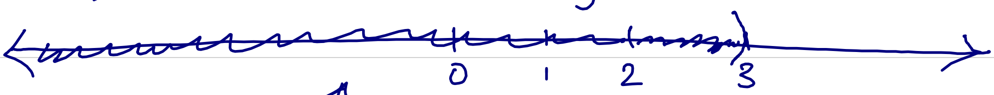
$$(0,1] = \{x \in \mathbb{R} : 0 < x \le 1\}$$

Note that 0 is not in this set, but 1 is in there.

x lies here

EXERCISE 8 $[0, 1) = ?$

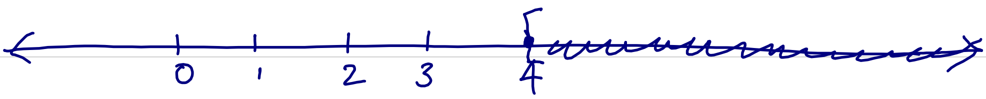
$(-\infty, 3) = \{x \in \mathbb{R} : \quad x < 3\}$



x lies here

$[4, \infty) = \{x \in \mathbb{R} : \quad 4 \le x\}$

Note 4 is included



Union of sets

Union of sets is the combination of sets. Formally, if A and B are sets,

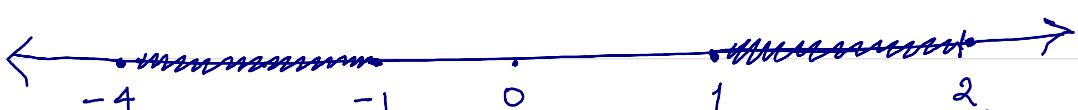
$A \cup B = \{$ set of elements in A "or" $B \}$
$= \{x : x \in A \text{ or } x \in B\}$.

Ex: $A = \{1, 3\}$, $B = \{1, 5, 7, \text{John}\}$

$A \cup B = \{1, 3, 5, 7, \text{John}\}$

This symol denotes 'union'.

EXERCISE 9



-4 not included

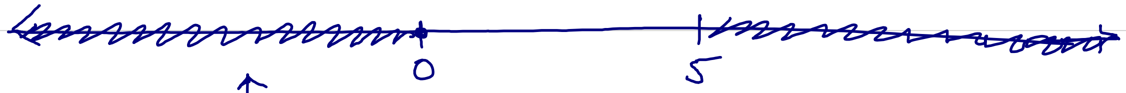
-1 included

1 not included

2 not included

Write the shaded set of $\mathbb{R}$ using interval notation.

EXERCISE 10



Everything on the left of 0 including 0

everything past this point not including 5

Express the above set using interval notation.

Finding domains of functions

The questions of this sort can be interpreted as asking what are the possible input values for which there is a valid output.

General strategy

$f(x) = \sqrt{x}$

Square roots are defined for real numbers that are greater than or equal to zero. What is $\sqrt{-1}$. There is no real solution. So Domain $= [0, \infty)$

$f(x) = \dfrac{1}{x}$. x cannot be 0 because $\dfrac{1}{0}$ is not defined. So Domain $= (-\infty, 0) \cup (0, \infty)$.

$f(x) = \sqrt[3]{x}$.

Cube roots are well defined for any real number. So Domain $= \mathbb{R}$ or $(-\infty, \infty)$.

In general for $f(x) = \sqrt[n]{x}$, Domain $= \begin{cases} \mathbb{R} & \text{if } n \text{ odd} \\ [0, \infty) & \text{if } n \text{ even} \end{cases}$

Problem

Find domain of $f(x) = \dfrac{\sqrt{x-1}}{\sqrt{9-2x}}$.

Solution. Numerator:

$$x - 1 \geq 0$$

$$\text{or,} \quad x \geq 1. \qquad (1)$$

Denominator..

$$9 - 2x \geq 0 \quad \text{for squareroot to be defined,}$$

$$\text{and} \quad 9 - 2x > 0 \quad \text{because denominator cannot be } 0.$$

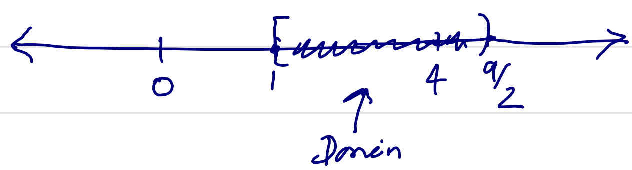
$$\text{Thus,} \quad 9 - 2x > 0$$

$$\text{or,} \quad 9 > 2x$$

$$\text{or,} \quad \frac{9}{2} > x. \qquad (2)$$

Thus, from (1) and (2),

$$\text{Domain} = \left[1, \frac{9}{2}\right)$$



EXERCISE 11 Find domain of

(a) $f(x) = \sqrt{x-3}$

(b) $g(x) = \dfrac{1}{x^2 - 4}$