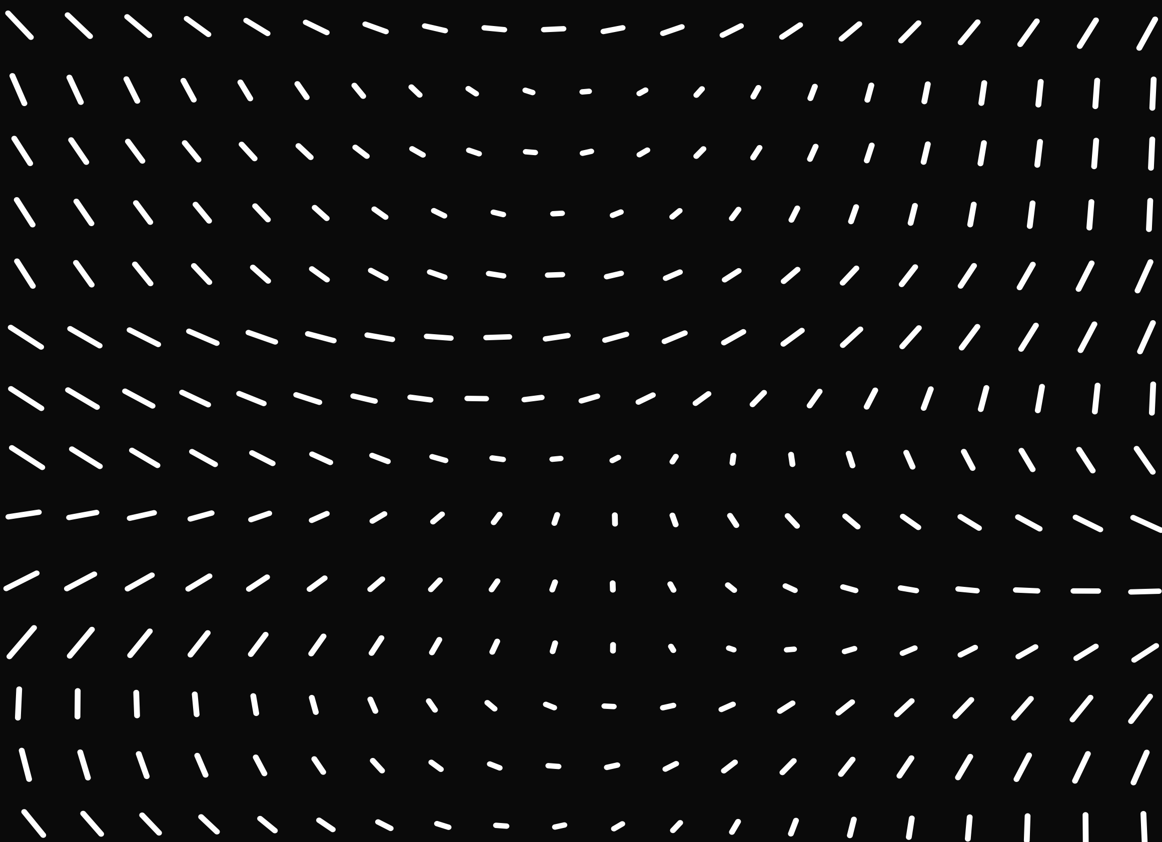
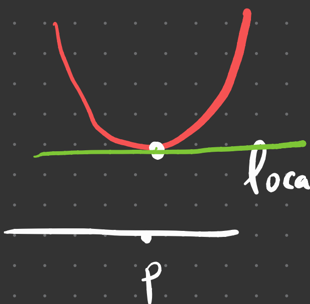


HW 8 Due Tuesday

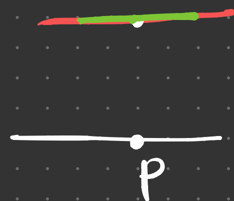
Midterm 3 April 1 Thursday

Goal: Find local minimums / maximums of functions.

Local Minimum:



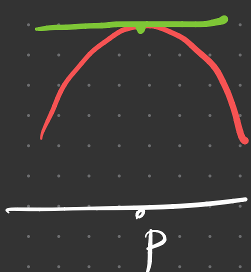
locally

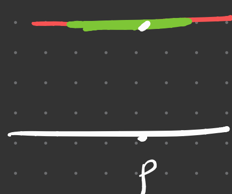


locally

$$f'(p) = 0$$

Local Maximum:





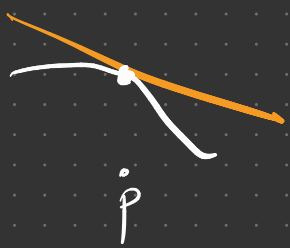
$$f'(p) = 0$$

Def. A point p is said to be a <u>critical point</u> if $f'(p) = 0$.

We observed that if p is a local minimum or local maximum, then p is a critical point, i.e. $f'(p) = 0$.

Ques. Is the converse true? If p is a critical point, i.e., $f'(p) = 0$, does it imply that p is a local minimum or local maximum?

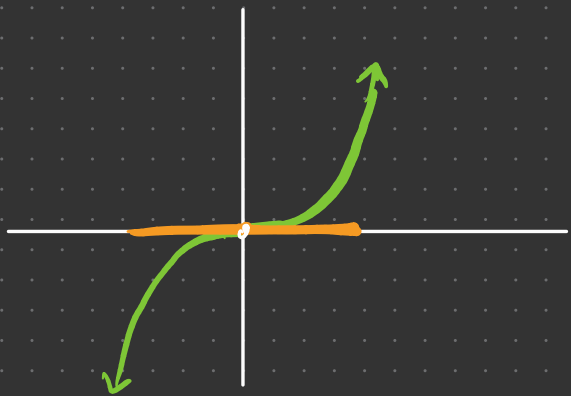
Ans. No.



$f'(p) < 0$

Not a critical point because $f'(p) \neq 0$.

$f(x) = x^3$



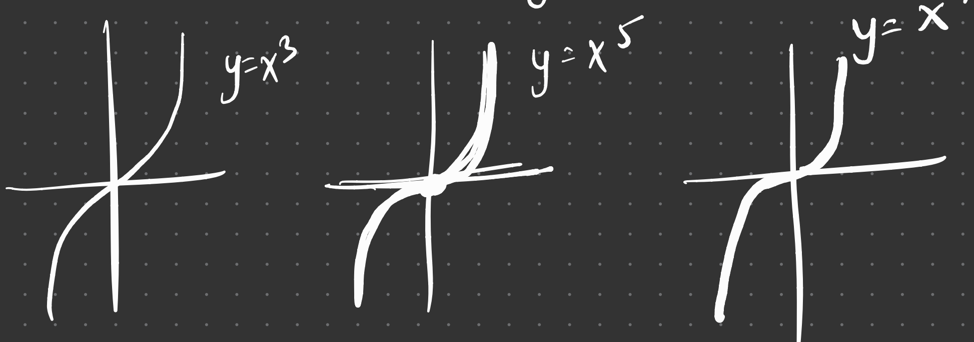
$$f(x) = x^3$$
$$f'(x) = 3x^2$$
$$f'(0) = 3 \cdot 0^2 = 0$$

0 is a critical point of this function.

But 0 is neither a local minimum nor a local maximum.

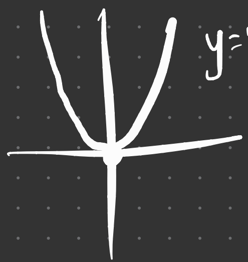
$$f(x) = x^n$$

n is odd and greater than 1.

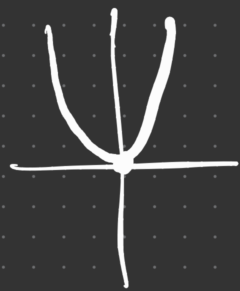


$f(x) = x^n$

n even



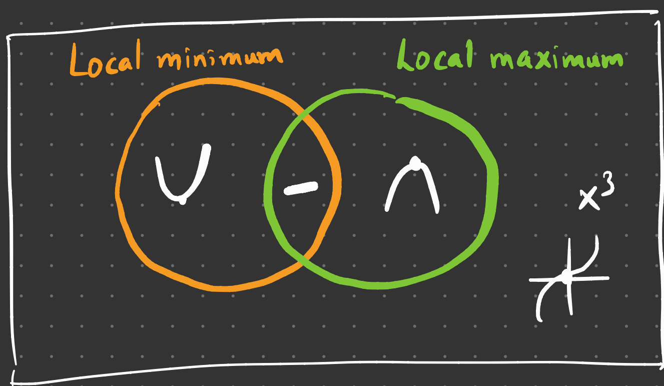
$y = x^2$

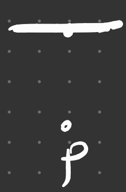


$y = x^4$

local minimum at 0

Critical points
$f'(p) = 0$





Strategy to find local minimum / maximum:

1. Find critical points, i.e., all points where the derivative is 0.

2. For each critical point use

    1. First derivative test

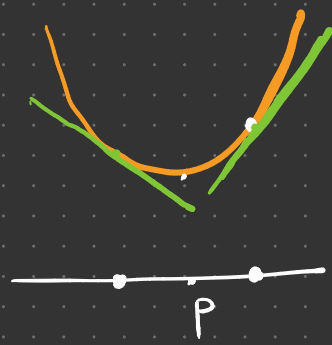
    OR

    2. Second derivative test

to find whether they are local minimum, local maximum or neither.

First derivative test

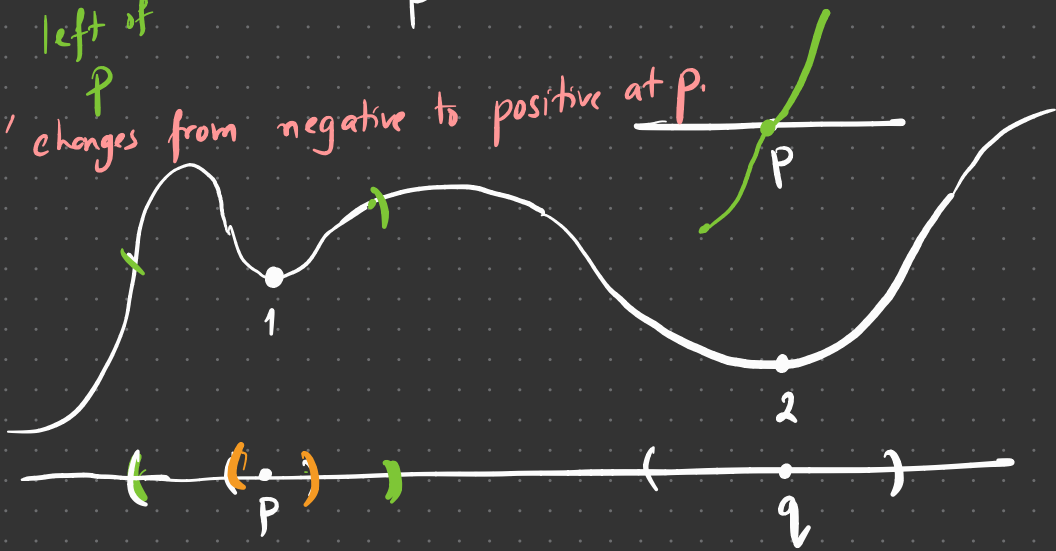
Local minimum:

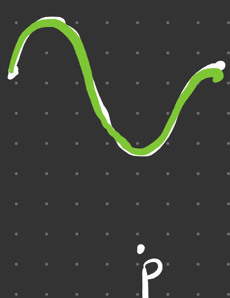


$f' < 0$ for points on the left of p

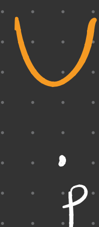
$f' > 0$ for points on the right of p.

f' changes from negative to positive at p.

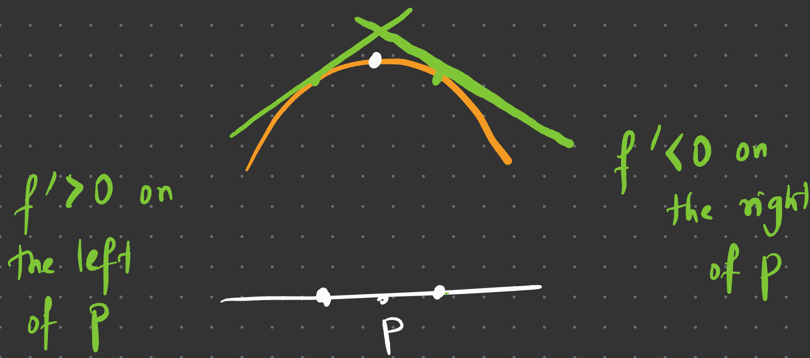


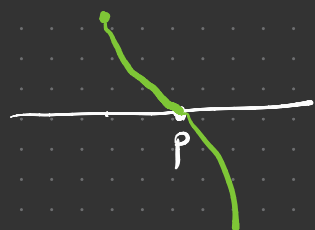


not a valley



Local maximum:





f' changes from positive to negative at p.

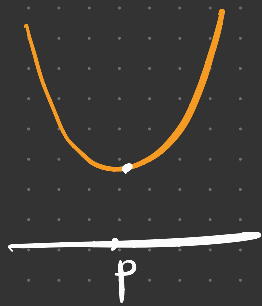
FIRST DERIVATIVE TEST

Suppose p is a critical point.

- If f' changes from negative to positive at p, then p is a local minimum.
- If f' changes from positive to negative at p, then p is a local maximum.

Second Derivative Test

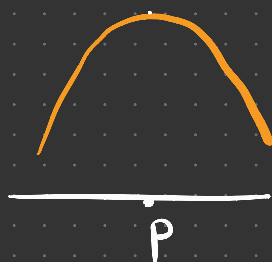
Local minimum:



concave up

A graph is concave up $\iff$ $f'' > 0$ on the interval on an interval

Local maximum:



concave down

A graph is concave down $\iff$ $f'' < 0$ on the interval. on an interval

Second Derivative Test

$f''(p) > 0$ Then f has a local minimum.

$f''(p) < 0$ then f has a local maximum.

$f''(p) = 0$, the test tells us nothing.

Example 1 $f(x) = x^3 - 9x^2 - 48x + 52$. Verify that f has a local maximum at $x = -2$ and a local minimum at $x = 8$.

Soln. First show they are critical points:

$$f'(x) = \frac{d}{dx}\left(x^3 - 9x^2 - 48x + 52\right)$$

$$= \frac{d}{dx}(x^3) - \frac{d}{dx}(9x^2) - \frac{d}{dx}(48x) + \frac{d}{dx}52$$

$$= 3x^2 - 18x - 48$$

$$f'(-2) = 3(-2)^2 - 18(-2) - 48$$

$$= 12 + 36 - 48$$

$$= 0$$

-2 is a critical point.

$$f'(8) = 3(8)^2 - 18\cdot(8) - 48$$

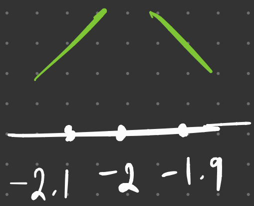
$$= 3\cdot 64 - 18\cdot 8 - 48 =$$

$$= 0$$

8 is a critical point.

Want: -2 is a local maximum

First derivative test



$$f'(-2.1) = 3(-2.1)^2 - 18(-2.1) - 48$$
$$= 3.03 > 0$$

$$f'(-1.9) = 3(-1.9)^2 - 18(-1.9) - 48$$
$$= -2.97 < 0$$

Thus, -2 is a local maximum.

Second derivative test

$$f'(x) = 3x^2 - 18x - 48$$
$$f''(x) = 6x - 18$$
$$f''(-2) = 6(-2) - 18 = -12 - 18 = -30 < 0$$

By the second derivative test, -2 is a local maximum

$x = 8$ **Exercise:**

Verify it is a local minimum.

Example 2 Find the critical points of the function and classify them as local max. or local min. or neither.

a) $g(x) = xe^{-3x}$

Soln. $g(x) = f(x)\,h(x)$

where $f(x) = x$ and $h(x) = e^{-3x}$

$f'(x) = 1$ and $h'(x) = -3e^{-3x}$

[Derivative of $e^{kx} = ke^{kx}$]

$$g'(x) = f'(x)h(x) + f(x)h'(x)$$

$$= 1 \cdot e^{-3x} + x \cdot (-3e^{-3x})$$

$$= e^{-3x} - 3xe^{-3x}$$

$$= (1 - 3x)e^{-3x}$$

Want: Critical points

Solve $g'(x) = 0$

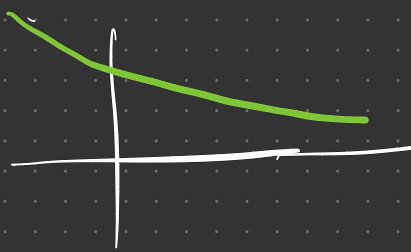
$$(1-3x)e^{-3x} = 0$$

$\Rightarrow \quad 1-3x = 0$

since e^{-3x} is always positive.

Note that

$$e^{-3x} > 0$$



$\Rightarrow \quad 3x = 1$

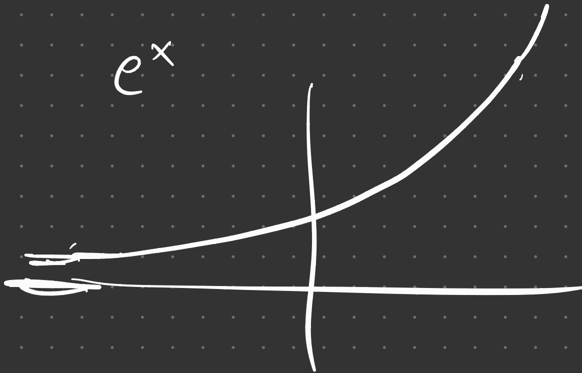
$\Rightarrow \quad x = 1/3$

First derivative test / Second deriv. test

Exercise.

⋮

e^x



$e^{\boxed{\ }}$

e^{-x}