HW 7 Due March 16

Bonus Assignment Due March 16

Problem Differentiate

$$y = 5\ln t + 7e^t - 4t^2 + 12$$

Soln. $\frac{d}{dt}\left(5\ln t + 7e^t - 4t^2 + 12\right)$

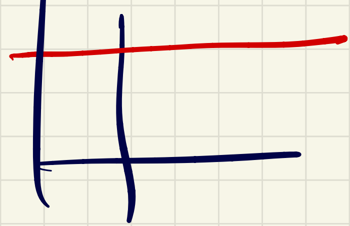
$$= \frac{d}{dt}(5\ln t) + \frac{d}{dt}(7e^t) - \frac{d}{dt}(4t^2) + \frac{d}{dt}12$$

$$= 5\frac{d}{dt}(\ln t) + 7\frac{d}{dt}(e^t) - 4\frac{d}{dt}t^2 + \frac{d}{dt}(12)$$

$$= 5\cdot\frac{1}{t} + 7e^t - 4\cdot 2t + 0$$

$$= \boxed{\frac{5}{t} + 7e^t - 8t}$$

$$\frac{d}{dt}k = 0$$



$$\frac{d}{dt}(4t^2)$$

$$= 4 \cdot \frac{d}{dt}(t^2)$$

$$= 4 \cdot 2t^{2-1}$$

$$= 4 \cdot 2t^1$$

$$= 8t$$

$$\left[\text{Power Rule:} \quad \frac{d t^n}{dt} = n t^{n-1} \right]$$

$; n = 2$

$$\frac{d}{dt}(5 \ln t)$$

$$= 5 \cdot \frac{d}{dt}(\ln t)$$

$$= \frac{5}{1} \cdot \frac{1}{t}$$

$$= \frac{5}{t}$$

$$\left[\frac{d}{dt} \ln t = \frac{1}{t} \right]$$

3.3 Chain Rule

This is used to find the derivative of composition of functions.

Recall: Let f and g be two functions.

$f \circ g$ (New function)

$$f \circ g(x) = f(g(x))$$

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[rr, bend right, "f \circ g"'] & g(x) \arrow[r, "f"] & f(g(x)) \end{tikzcd}$$

$g \circ f$ (New function)

$$g \circ f(x) = g(f(x))$$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & f(x) \arrow[r, "g"] & g(f(x)) \end{tikzcd}$$

Goal: Find derivative of $f \circ g$ ($g \circ f$) using the derivatives of f and g.

Ex $f(x) = x^5$, $g(x) = x^2+1$

$$f \circ g(x) = f(g(x))$$
$$= f(x^2+1)$$
$$= (x^2+1)^5$$

$$g \circ f(x) = g(f(x))$$
$$= g(x^5)$$
$$= (x^5)^2 + 1$$
$$= x^{10} + 1$$

Chain Rule

Let $f(x)$, $g(x)$ be two functions.

$$\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)}$$

Written in different notation:

$$y = f(z)$$

$$z = g(t)$$

Same thing

$$\boxed{\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}}$$

$$t \xmapsto{g} z \xmapsto{f} y$$

We Know:

1> constant $f(x) = k$

2> Linear $f(x) = mx + b$

3> Power $f(x) = x^n$

4> Polynomials $f(x) = a_n x^n + \cdots + a_0$

5. Exponential $f(x) = a^x$

6. Logarithm $f(x) = \ln x$

Problem 1

Differentiate

a) $(3t^3 - t)^5$

Soln.

$$t \xmapsto{g} 3t^3 - t \xmapsto{f} (3t^3 - t)^5$$

$$g(t) = 3t^3 - t$$

$$f(t) = t^5$$

$$(3t^3 - t)^5 = f(g(t))$$

Power Rule

$$\frac{d}{dt} x^n = n x^{n-1}$$

$$n = 5$$

By Chain Rule,

$$\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t)$$

$$f'(t) = 5t^{5-1} = 5t^4$$

$$f'(g(t)) = f'(3t^3 - t)$$

$$= 5(3t^3 - t)^4$$

$$g'(t) = \frac{d}{dt}(3t^3 - t)$$

$$= \frac{d}{dt}(3t^3) - \frac{d}{dt} t = 9t^2 - 1$$

$$\boxed{\frac{d}{dt} f(g(t)) = 5(3t^3 - t)^4 \cdot (9t^2 - 1)}$$

Alternative notation:

$$\frac{d}{dt}(3t^3 - t)^5$$

Let $z = 3t^3 - t$

$$\frac{d}{dt}(3t^3 - t)^5 = \frac{d}{dt} z^5$$

$$= \frac{d}{dz} z^5 \cdot \frac{dz}{dt} \quad \text{[Chain Rule]}$$

$$= 5z^{5-1} \cdot \frac{d}{dt}(3t^3 - t)$$

$$= 5z^4 \cdot (9t^2 - 1)$$

$$= \boxed{5(3t^3 - t)^4 (9t^2 - 1)}$$

b) $\ln(q^2+1)$

Soln.

$$q \xmapsto{g} q^2+1 \xmapsto{f} \ln(q^2+1)$$

$$g(q) = q^2+1$$

$$f(q) = \ln(q)$$

Thus, $\ln(q^2+1) = f(g(q))$

By Chain Rule,

$$\frac{d}{dq} f(g(q)) = f'(g(q)) \cdot g'(q)$$

$$f'(q) = \frac{d}{dq} \ln q = \frac{1}{q}$$

$$f'(g(q)) = f'(q^2+1) = \frac{1}{q^2+1}$$

$$g'(q) = \frac{d}{dq}(q^2+1) = 2q^{2-1} + 0$$

$$= 2q$$

$$\therefore \frac{d}{dq} \ln(q^2+1) = \frac{1}{q^2+1} \cdot 2q = \boxed{\frac{2q}{q^2+1}}$$

Alternative notation:

$$\frac{d}{dq} \ln(q^2+1)$$

Let $z = q^2+1$.

$$\frac{d}{dq} \ln(q^2+1) = \frac{d}{dq} \ln(z)$$

$$= \frac{d}{dz} \ln(z) \cdot \frac{dz}{dq} \quad \text{(Chain Rule)}$$

$$= \frac{1}{z} \cdot \frac{d}{dq}(q^2+1)$$

$$= \frac{1}{z} \cdot (2q+0)$$

$$= \frac{1}{z} \cdot 2q$$

$$= \frac{1}{q^2+1} \cdot \frac{2q}{1}$$

$$= \boxed{\frac{2q}{q^2+1}}$$

c) e^{-x^2}

First method:

$$x \xmapsto{g} -x^2 \xmapsto{f} e^{-x^2}$$

$$g(x) = -x^2$$
$$f(x) = e^x$$

Hence, $e^{-x^2} = f(g(x))$

By Chain Rule,

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$f'(x) = \frac{d}{dx} e^x = e^x$$

$$\Rightarrow f'(g(x)) = f'(-x^2) = e^{-x^2}$$

$$g'(x) = \frac{d}{dx}(-x^2) = -2x^{2-1} = -2x$$

$$\therefore \frac{d}{dx} f(g(x)) = e^{-x^2} \cdot (-2x)$$
$$= \boxed{-2xe^{-x^2}}$$

Second Method:

$$\frac{d}{dx} e^{-x^2}$$

Let $z = -x^2$.

$$\frac{d}{dx} e^{-x^2} = \frac{d}{dx} e^{z}$$

$$= \frac{d}{dz} e^{z} \cdot \frac{dz}{dx}$$ [Chain Rule]

$$= e^{z} \cdot \frac{d}{dx}(-x^2)$$

Powe Rule

$n=2$

$$= e^{z} \cdot (-2x)$$

$$= e^{-x^2} (-2x)$$

$$= \boxed{-2xe^{-x^2}}$$

c) $\sqrt{1+2e^{5t}}$

<u>Soln.</u> $\frac{d}{dt}\sqrt{1+2e^{5t}} = ?$

Let $z = 1+2e^{5t}$.

$$\frac{d}{dt}\sqrt{1+2e^{5t}} = \frac{d}{dt}\sqrt{z}$$

$$= \frac{d}{dz}\sqrt{z} \cdot \frac{dz}{dt} \quad \text{(Chain Rule)}$$

Power Rule

$$= \frac{d}{dz} z^{\frac{1}{2}} \cdot \frac{d}{dt}\left(1+2e^{5t}\right)$$

$\frac{d}{dt}e^{kt} = ke^{kt}$

$$= \frac{1}{2} \cdot z^{\frac{1}{2}-1} \cdot \left(\cancel{\frac{d}{dt}(1)} + \frac{d}{dt}2e^{5t}\right)$$

0

$$= \frac{1}{2} z^{-\frac{1}{2}} \cdot \left(2 \cdot 5 \cdot e^{5t}\right)$$

$z^{-1/2} = \frac{1}{z^{1/2}}$

$= \frac{1}{\sqrt{z}}$

$$= \frac{1}{2z^{1/2}} \cdot 10e^{5t}$$

$$= \boxed{\frac{5e^{5t}}{\sqrt{1+2e^{5t}}}}$$