Lie Algebras

- GOAL OF REPRESENTATION THEORY IS TO CLASSIFY AND STUDY THE IRREDUCIBLE REPRESENTATIONS OF LIE GROUPS, ALGEBAIC GROUPS, LIE ALGEBRAS ETC.

- GOAL OF GEOMETRIC REPRESENTATION THEORY IS TO GIVE A GEOMETRIC INTERPRETATION OF THOSE REPRESENTATIONS.

BOREL WEIL BOTT

$$\left\{\begin{array}{c}\text{finite dim. irred.}\\ \text{Representations of}\\ G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}G\text{-equivariant}\\ \text{line bundles over}\\ \text{flag variety } G/B\end{array}\right\}\Big/W$$

BEILINSON BERNSTEIN LOCALIZATION

$$\left\{\begin{array}{c}\text{Respresentations of}\\ \mathfrak{g}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}D\text{-modules on}\\ G/B\end{array}\right\}$$

KIRILLOV's ORIBIT METHOD

$$\left\{\begin{array}{c}\text{Representations of}\\ G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{coadjoint orbits}\\ \text{of } G \text{ which are}\\ \text{symplectic}\end{array}\right\}$$

G is algebraic / Lie. B is Borel subgroup (maximal solvable subgroup)

Plan: finish algebraic groups

Borel-Bott-Weil

D-Modules

Springer correspondence

Hecke Algebras

Geometric Satake

Kazhdan-Lustig

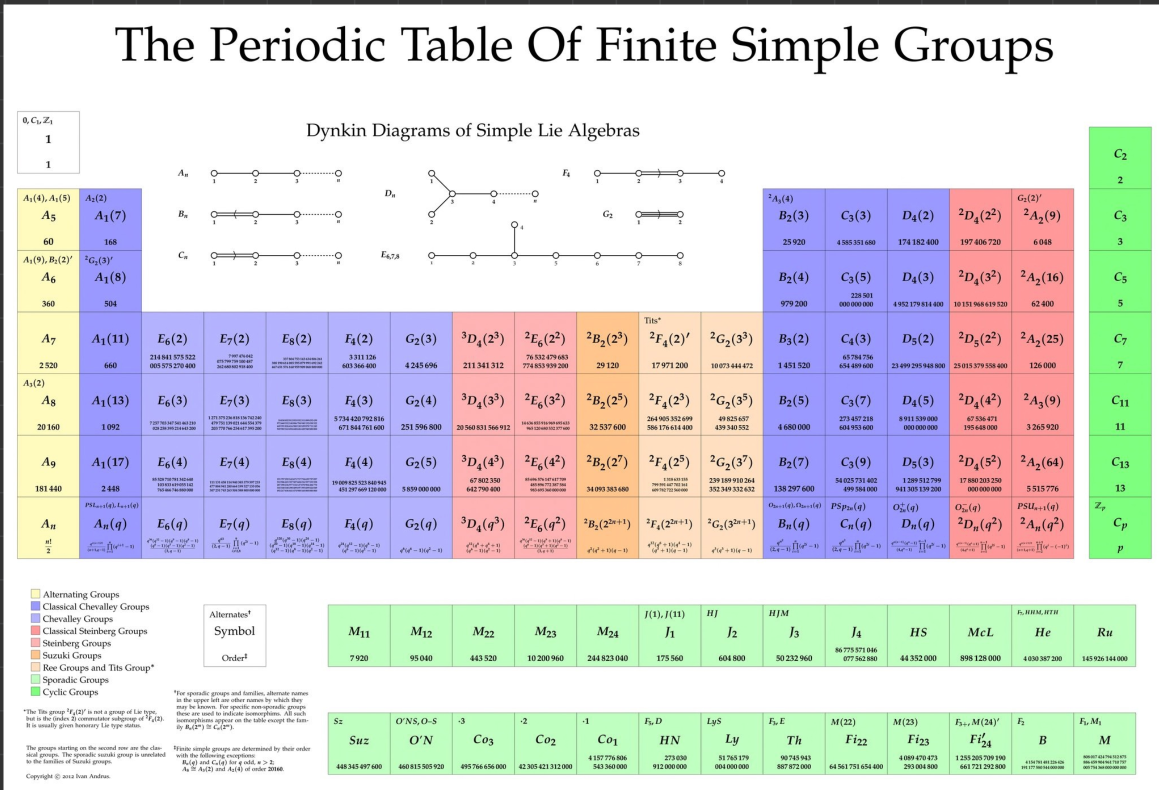
Perverse sheaves

Quiver varieties

Today: Basics Lie Algebras

CLASSIFICATION OF OBJECTS AND THEIR REPRESENTATIONS

- Classification



Why classify simple groups?

Because every group is built out of a composition series:

$$1 = G_0 \triangleleft G_1 \ldots \triangleleft G_n = G$$

with G_i / G_{i-1} simple. Jordan-Hölder

- Representations

The Character Table of the symmetric group S_4

	1	6	8	3	6
	1	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	3	1	0	−1	−1
χ_4	3	−1	0	−1	1
χ_5	2	0	−1	2	0

Irreducible representations are enumerated by character tables.

Why classify the irreducible representations?

Because every representation over characterstic 0 is built out of finite dimensional representations!

Thm. (Masche). Let $\rho : G \to GL(V)$ be a linear representation where V is a vector space over a field of char 0. Let W be a G-invariant subspace of V. Then the complement $W^\perp$ of W exists in V and is G-invariant.

COMPACT LIE GROUPS

Classification

First Reduction: Let G be a compact Lie group, i.e. a smooth manifold with a group structure. Take its connected identity component G_0. Then

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow \pi_0(G) \longrightarrow 1$$

(connected components)

$\pi_0(G)$ is finite because G is compact.

So, it suffices to study connected compact Lie groups.

Examples:

- $\mathbb{R}^\times$ 

$$1 \rightarrow \mathbb{R}_{>0} \rightarrow \mathbb{R}^\times \rightarrow \mathbb{Z}_2 \rightarrow 1$$

- $U(2)$

$$1 \rightarrow SU(2) \rightarrow U(2) \rightarrow \mathbb{Z}_2 \rightarrow 1$$

Second Reduction

Theorem: Every connected compact Lie group is the quotient by a finite central subgroup of a product of a simply connected compact Lie group and a torus.

Reduces the problem of classification to simply connected Lie groups and their centers.

$(S^1)^n$

Linearization

If G is a compact connected simply connected Lie group, the complexification of the Lie algebra of G is semisimple. Conversely, every semisimple Lie algebra has a compact real form isomorphic to the Lie algebra of a compact, simply connected Lie group.

Remark: There is a natural functor

$$\text{LieGrp.} \longrightarrow \text{LieAlg.}$$

In the case of simply connected Lie groups we have:

$$\text{LieGrp}_{\text{simply conn.}} \simeq \text{Lie Alg.}$$

Dynkin

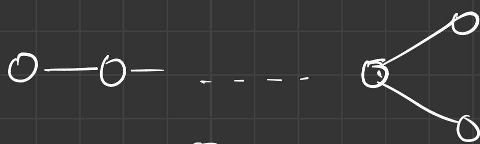
Simple Lie algebras are classified by their root systems which in turn are classified by Dynkin diagrams.

A_n:

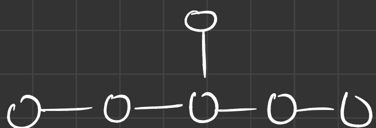
B_n:

C_n:

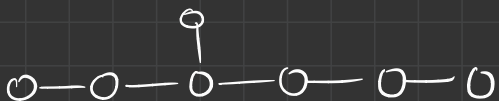
D_n:



E_6:



E_7:



E_8:

F_4:

G_2

From this we can classify the simply connected compact groups:

- $SU(n) \longleftrightarrow A_{n-1}$
- $Spin(2n+1) \longleftrightarrow B_n$
- $Sp(n) \longleftrightarrow C_n$
- $Spin(2n) \longleftrightarrow D_n$
- $\longleftrightarrow G_2$
- $\longleftrightarrow F_4$
- $\longleftrightarrow E_6$
  $\longleftrightarrow E_7$
- $\longleftrightarrow E_8$

Representations

The irreducible representations are given by the irreducible highest weight modules for dominant integral weights.

Lie Algebras

Def. A Lie algebra $\mathfrak{g}$ over a field k is a vector space over k with a k-bilinear map $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies

(i) (skew-symmetric) $[x,y] = -[y,x]$

(ii) (Jacobi Identity) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

$$\Updownarrow$$

$$\operatorname{ad}[x,y] = \operatorname{ad}x \operatorname{ad}y - \operatorname{ad}y \operatorname{ad}x$$

Here $\operatorname{ad}x : \mathfrak{g} \to \mathfrak{g}$ is defined by $y \mapsto [x,y]$

Examples: Classical groups

$\mathfrak{gl}(n,\mathbb{C}) = \mathrm{Mat}_{n\times n}(\mathbb{C})$

$\mathfrak{sl}(n,\mathbb{C})$

$\mathfrak{su}(n,\mathbb{C})$

$\mathfrak{u}(n,\mathbb{C})$

$\mathfrak{sp}(n,\mathbb{C})$

$[\,,\,]$ is $[A,B] = AB - BA$

Simple $\subset$ Semisimple $\supset$ Solvable $\supset$ Nilpotent $\supset$ Abelian

<u>Def</u> A Lie algebra $\mathfrak{g}$ is <u>abelian</u> if $[x, y] = 0 \ \forall x, y \in \mathfrak{g}$.

All of them are $\mathbb{C}^n$ or $\mathbb{R}^n$ with trivial commutator.

<u>Def.</u> A Lie algebra $\mathfrak{g}$ is <u>nilpotent</u> if

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \supseteq \cdots\cdots \{0\}$$

Define $D_0 \mathfrak{g} = \mathfrak{g}$ and

$$D_{i+1} = [\mathfrak{g}, D_i \mathfrak{g}]$$

The above condition says $D_n \mathfrak{g} = 0$ for some n.

<u>Example</u> $\begin{pmatrix} 0 & & * \\ & 0 & \\ & & 0 \end{pmatrix}$ strictly upper triangular matrices

<u>Def.</u> A Lie algebra $\mathfrak{g}$ is <u>solvable</u> if

$$\mathfrak{g} \supseteq [\mathfrak{g}, \mathfrak{g}] \supseteq [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]] \supseteq \cdots \{0\}$$

Define $D^0 \mathfrak{g} = \mathfrak{g}$ and

$$D^{i+1} \mathfrak{g} = [D^i \mathfrak{g}, D^i \mathfrak{g}]$$

The above condition says $D^i \mathfrak{g} = 0$ for some n.

<u>Example</u> $\begin{pmatrix} * & & * \\ & * & \\ & & * \end{pmatrix}$ upper-triangular matrices

Simple) Semisimple) (Solvable (Nilpotent (Abelian

Semisimple Lie algebras are on the opposite side of the spectrum. They are as far as possible from being abelian.

Def A Lie algebra $\mathfrak{g}$ is called <u>semisimple</u> if it contains no nonzero solvable ideal.

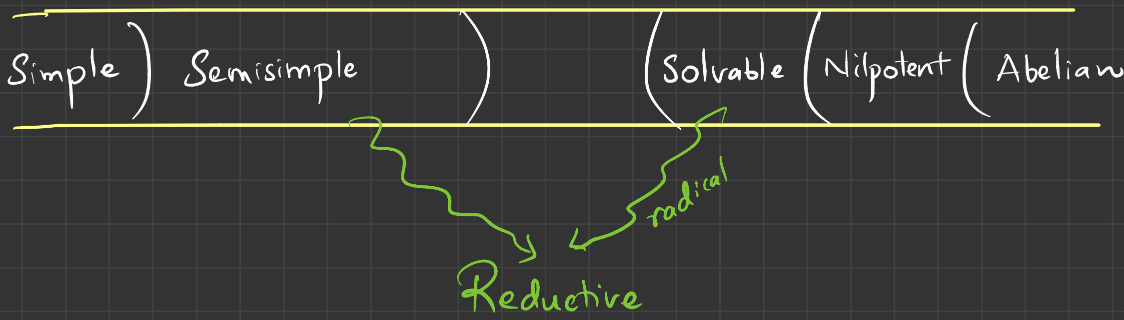
Note that this implies that the center $\mathfrak{z}(\mathfrak{g}) = 0$.

Def A Lie algebra $\mathfrak{g}$ is called <u>simple</u> if it is not abelian and contains no ideals other than 0 and $\mathfrak{g}$.

Examples: $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$.

Lemma Any simple Lie algebra is semisimple.

Proof. If $\mathfrak{g}$ is simple, then it contains no ideals other than 0 and $\mathfrak{g}$. Thus, if $\mathfrak{g}$ contains a nonzero solvable ideal, then it must coincide with $\mathfrak{g}$, so $\mathfrak{g}$ must be solvable. But $[\mathfrak{g}, \mathfrak{g}]$ is an ideal which is strictly smaller than $\mathfrak{g}$ because $\mathfrak{g}$ is solvable and nonzero because $\mathfrak{g}$ is not abelian. This gives a contradiction. $\square$



Theorem In any Lie algebra $\mathfrak{g}$, there is a unique solvable ideal which contains any other solvable ideal. This solvable ideal is called the <u>radical</u> of $\mathfrak{g}$ and is denoted by $\operatorname{rad}(\mathfrak{g})$.

<u>Proof</u>. If I_1 and I_2 are solvable ideals then so is $I_1 + I_2$. $I_1 + I_2$ contains solvable ideal I_1 and the quotient $(I_1 + I_2)/I_1 = I_2/(I_1 \cap I_2)$ is also solvable since it is a quotient of I_2. Thus by a lemma, $I_1 + I_2$ is also solvable. By induction, any finite sum of solvable ideals is solvable. Thus,

$$\operatorname{rad}(\mathfrak{g}) = \sum_{I \text{ solvable}} I$$

Finite-dimensionality of $\mathfrak{g}$ shows it suffices to take finite sum. $\square$

Using the definition above, we see $\mathfrak{g}$ is semisimple iff $\mathrm{rad}(\mathfrak{g}) = 0$.

__Theorem__ For any Lie algebra $\mathfrak{g}$, $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple.

__Proof.__ Assume that $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ contains a solvable ideal I. Consider the ideal $\tilde{I} = \pi^{-1}(I) \subset \mathfrak{g}$, where $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$. Then $\tilde{I} \supset \mathrm{rad}(\mathfrak{g})$ and $\tilde{I}/\mathrm{rad}(\mathfrak{g}) \cong I$ is solvable. Thus, $\tilde{I}$ is solvable. So $\tilde{I} = \mathrm{rad}(\mathfrak{g})$. $\Rightarrow I = 0$. $\square$

__Theorem (Levi)__ Any Lie algebra can be written as

$$\mathfrak{g} = \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$$

where $\mathfrak{g}_{ss}$ is a semisimple subalgebra in $\mathfrak{g}$.

Example. $G = SO(3,\mathbb{R}) \ltimes \mathbb{R}^3$ Poincare group

$\mathfrak{g} = \mathfrak{so}(3,\mathbb{R}) \ltimes \mathbb{R}^3$

$[(A_1, b_1), (A_2, b_2)] = ([A_1, A_2], A_1 b_2 - A_2 b_1)$

$\mathbb{R}^3$ is abelian $\Rightarrow$ solvable ; $\mathfrak{so}(3,\mathbb{R})$ is semisimple.

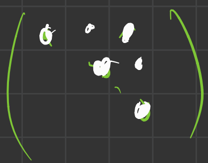
Simple) Semisimple) (Solvable (Nilpotent (Abelian

Reductive

Def. A Lie algebra is called <u>reductive</u> if $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$,

i.e. if $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple.

($\mathfrak{z}(\mathfrak{g})$ is the center of $\mathfrak{g}$)



Killing form and Cartan's criterion

Def. The Killing form is the bilinear form on $\mathfrak{g}$ defined by $K(x,y) = tr(ad x\, ad y)$

Theorem (Cartan) A Lie algebra $\mathfrak{g}$ is solvable iff $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$ (i.e. $K(x,y)=0$ for any $x \in [\mathfrak{g},\mathfrak{g}]$, $y \in \mathfrak{g}$).

Theorem (Cartan) A Lie algebra is semisimple iff the Killing form is non-degenerate.

$$ad x: \mathfrak{g} \longrightarrow \mathfrak{g}$$

$$ad x \cdot y = [x,y]$$

invariant bilinear form:

$$B: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

$$B(ad x \cdot y, z) + B(y, ad x \cdot z) = 0$$

Root Decomposition

<u>Def.</u> An element $x \in \mathfrak{g}$ is called <u>semisimple</u> if $\mathrm{ad}\, x$ is a semisimple operator $\mathfrak{g} \to \mathfrak{g}$.

An element $x \in \mathfrak{g}$ is called <u>nilpotent</u> if $\mathrm{ad}\, x$ is a nilpotent operator $\mathfrak{g} \to \mathfrak{g}$.

<u>Theorem</u> If $\mathfrak{g}$ is a semisimple complex Lie algebra, then any $x \in \mathfrak{g}$ can be uniquely written in the form

$$x = x_s + x_n$$

where x_s is semisimple, x_s is nilpotent, and $[x_s, x_n] = 0$.

<u>Proof.</u> Uniqueness follows from uniqueness of the Jordan decomposition for $\mathrm{ad}\, x$: If

$$x = x_s + x_n = x_s' + x_n'$$

Then $(\mathrm{ad}\, x)_s = \mathrm{ad}\, x_s = \mathrm{ad}\, x_s'$, so $\mathrm{ad}(x_s - x_s') = 0$.

Since a semisimple Lie algebra has zero center, $x_s - x_s' = 0$.

To prove existence, write $\mathfrak{g}$ as a direct sum of generalized eigenspaces for $\mathrm{ad}\, x$: $\mathfrak{g} = \bigoplus \mathfrak{g}_\lambda$,

$$(\mathrm{ad}\, x - \lambda\, \mathrm{id})^n \big|_{\mathfrak{g}_\lambda} = 0 \quad \text{for } n \gg 0.$$

Def. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called <u>toral</u> if it is commutative and consists of semisimple elements.

<u>Theorem</u> Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a toral subalgebra, and let $(\,,\,)$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (for example, the Killing form). Then

1) $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is a common eigenspace for all operators $\operatorname{ad} h$, $h \in \mathfrak{h}$, with eigenvalue α:

$$\operatorname{ad} h.\, x = \langle \alpha, h \rangle x \,, \quad h \in \mathfrak{h},\, x \in \mathfrak{g}_\alpha.$$

In particular, $\mathfrak{h} \subset \mathfrak{g}_0$.

2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

3) If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are orthogonal wrt $(\,,\,)$.

4) For any α, the form $(\,,\,)$ gives a non-degenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$.

Theorem Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a toral subalgebra, and let $(,)$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (for example, the Killing form). Then

1) $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha$ is a common eigenspace for all operators $\operatorname{ad} h$, $h \in \mathfrak{h}$, with eigenvalue α:
$$\operatorname{ad} h . x = \langle \alpha, h \rangle x \ , \ h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha .$$
In particular, $\mathfrak{h} \subset \mathfrak{g}_0$.

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3) If $\alpha + \beta \neq 0$, then $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are orthogonal wrt $(,)$.

4) For any α, the form $(,)$ gives a non-degenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \longrightarrow \mathbb{C}$.

Proof. By definition, for each $h \in \mathfrak{h}$, $\operatorname{ad} h$ is diagonalizable. Since all operators $\operatorname{ad} h$ commute, they can be simultaneously diagonalized. This proves 1).

$$\operatorname{ad} h . [y, z] = [\operatorname{ad} h . y, z] + [y, \operatorname{ad} h z]$$
$$= \langle \alpha, h \rangle [y, z] + \langle \beta, h \rangle [y, z]$$
$$= \langle \alpha + \beta, h \rangle [y, z] \quad \text{(Proves 2))}$$

If $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $h \in \mathfrak{h}$, then by invariance of $(,)$,
$$([h, x], y) + (x, [h, y]) = (\langle \alpha, h \rangle + \langle \beta, h \rangle)(x, y) = 0.$$
Thus if $(x, y) \neq 0$, then $\langle \alpha + \beta, h \rangle = 0 \ \forall \ h \in \mathfrak{h}$ which implies $\alpha + \beta = 0$

The final part follows from non-degeneracy of $(,)$.

Cartan subalgebra

Def. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. A Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a toral subalgebra which coincides with its centralizer: $C(\mathfrak{h}) = \{ x \mid [x, \mathfrak{h}] = 0 \} = \mathfrak{h}$

Example: Let $\mathfrak{g} = sl(n, \mathbb{C})$ and $\mathfrak{h} = \{$diagonal matrices with trace $0\}$

$\mathfrak{h}$ is Cartan subalgebra. It is commutative, and every diagonal element is semisimple, so it is a toral subalgebra. Choose $h \in \mathfrak{h}$ to be a diagonal matrix with distinct eigenvalues. We know that if $[x, h] = 0$ and h has distinct eigenvalues, then any eigenvector of h is also an eigenvector of x; thus x must be diagonal. Thus, $C(\mathfrak{h}) = \mathfrak{h}$.

Existence

Theorem Let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal toral subalgebra, i.e. a toral subalgebra which is not properly contained in any other toral subalgebra. Then $\mathfrak{h}$ is a Cartan subalgebra.

Root decomposition:

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Thm. 1) $$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha = \{ x \mid [h, x] = \langle \alpha, h \rangle x \ \forall h \in \mathfrak{h} \}$

$$R = \{ \alpha \in \mathfrak{h}^* - \{0\} \mid \mathfrak{g}_\alpha \neq 0 \}$$

R is called root system of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ root subspaces.

2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ $\quad (\mathfrak{g}_0 = \mathfrak{h})$

3) If $\alpha + \beta \neq 0$, $\mathfrak{g}_\alpha, \mathfrak{g}_\beta$ are orthogonal

4) For any α, the Killing form gives a non-degenerate pairing $\mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha} \to \mathbb{C}$. In particular, restriction of K to $\mathfrak{h}$ is non-degenerate.

$\mathfrak{sl}(2,\mathbb{C})$

$$\mathfrak{sl}(2,\mathbb{C}) = \{ X \in \mathrm{Mat}(2,\mathbb{C}) \mid \operatorname{tr} X = 0 \}$$

This is the Lie algebra of $SL(2,\mathbb{C})$.

The bracket $[\ ,\ ]$ is just the usual bracket in $\mathrm{Mat}(2,\mathbb{C})$

$[A, B] = AB - BA$.

$$SL(2,\mathbb{C}) = \{ A \in \mathrm{Mat}(2,\mathbb{C}) \mid \det A = 1 \}$$

Note that $\mathfrak{sl}(2,\mathbb{C})$ has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Killing, Cartan say study the adjoint action.

$\mathfrak{g}$ Lie algebra.

$$\begin{aligned} \operatorname{ad} x : \mathfrak{g} &\longrightarrow \mathfrak{g} \\ y &\longmapsto [x,y] \end{aligned}$$

, i.e. $\operatorname{ad} x \, y = [x,y]$

$$[e,f] = \left[\begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\right]$$

$$= \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0&0\\1&0\end{pmatrix} - \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&0\end{pmatrix} - \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&-1\end{pmatrix} = h$$

$$\therefore [e,f] = h$$

$$\therefore [h,e] = \left[\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}\right]$$

$$= \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix} - \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$= 2e$$

$$\therefore [h,f] = -2f$$

Look at the operator $\operatorname{ad} h$. It is diagonalizable as $\operatorname{ad} h \cdot h = 0$, $\operatorname{ad} h \cdot e = 2e$, $\operatorname{ad} h \cdot f = -2f$. Its eigenspaces are $\mathbb{C}h$, $\mathbb{C}e$, $\mathbb{C}f$ with eigenvalues $0, 2, 2$, respectively.

$$\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f$$

Cartan

root eigenspaces.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

We can write $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ can be written E_{11}

and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ as E_{21}

$$\operatorname{ad} h \cdot E_{11} = (\epsilon_1 - \epsilon_2)\, E_{11} = (1-(-1))\, E_{11} = 2E_{11}$$

$$\operatorname{ad} h \cdot E_{21} = (\epsilon_2 - \epsilon_1)\, E_{21} = (-1-1)\, E_{21} = -2E_{21}$$

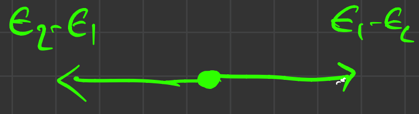
$$\epsilon_1 : \mathfrak{h} \to \mathbb{C}$$
$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \mapsto \lambda_1$$

$$\epsilon_2 : \mathfrak{h} \to \mathbb{C}$$
$$\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \to \lambda_2$$

The roots are $\epsilon_1 - \epsilon_2$ and $\epsilon_2 - \epsilon_1$ in $\mathfrak{h}^*$.

Root system of $\mathfrak{sl}(2,\mathbb{C})$

Type A_1



Dynkin Diagram

$\mathfrak{sl}(3,\mathbb{C})$

$$\mathfrak{sl}(3,\mathbb{C}) = \{ X \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{tr} X = 0 \}$$

Basis:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad f_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We can check that

- $[e_1, f_1] = h_1$; $[e_2, f_2] = h_2$, $[e_i, f_j] = 0 \quad i \neq j$

- $\operatorname{ad} h_1 \cdot e_1 = 2e_1 \qquad \operatorname{ad} h_1 e_2 = e_2 \qquad \operatorname{ad} h_1 e_3 = -e_3$

  $= (\epsilon_1 - \epsilon_2)(h_1)\, e_1 \qquad = (\epsilon_1 - \epsilon_3)(h_1) e_2 \qquad = (\epsilon_2 - \epsilon_3)(h_1)\, e_3$

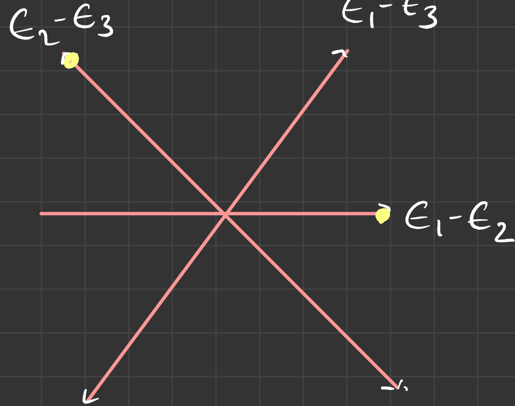
  $\operatorname{ad} h_1 \cdot f_1 = -2f_1 \qquad \operatorname{ad} h_1 f_2 = -f_2 \qquad \operatorname{ad} h_1 f_3 = f_3$

  $= (\epsilon_2 - \epsilon_1)(h_1)\, f_1 \qquad = (\epsilon_3 - \epsilon_1)(h_1)\, f_2 \qquad = (\epsilon_3 - \epsilon_2)(h_1)\, f_3$

- $\operatorname{ad} h_2 \cdot e_1 = -e_1 \qquad \operatorname{ad} h_2 e_2 = e_2 \qquad \operatorname{ad} h_2 e_3 = 2e_3$

  $\operatorname{ad} h_2 \cdot f_1 = f_1 \qquad \operatorname{ad} h_2 f_2 = -f_2 \qquad \operatorname{ad} h_2 f_3 = -2f_3$

Note that roots are $\epsilon_i - \epsilon_j \quad i \neq j$



Dynkin diagram:

Root decomposition:

Let $\mathfrak{h} = \langle h_1, h_2 \rangle$. $\mathfrak{h}$ is Cartan subalgebra.

$$\mathfrak{sl}(2,\mathbb{C}) = \underbrace{\mathfrak{g}_{\epsilon_2-\epsilon_1} \oplus \mathfrak{g}_{\epsilon_3-\epsilon_1} \oplus \mathfrak{g}_{\epsilon_3-\epsilon_2}}_{\mathfrak{n}_-} \oplus \mathfrak{h} \oplus \underbrace{\mathfrak{g}_{\epsilon_2-\epsilon_3} \oplus \mathfrak{g}_{\epsilon_1-\epsilon_3} \oplus \mathfrak{g}_{\epsilon_1-\epsilon_2}}_{\mathfrak{n}_+}$$

Triangular decomposition: $\boxed{\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+}$

<u>Example</u> $\mathfrak{g} = sl(n, \mathbb{C})$

$\mathfrak{h}$ = diagonal matrices with trace 0

Denote by $e_i : \mathfrak{h} \to \mathbb{C}$ the functional which computes i^{th} diagonal entry of h:

$$\begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix} \longmapsto h_i$$

Then $\sum e_i = 0$, so

$$\mathfrak{h}^* = \bigoplus \mathbb{C} e_i \Big/ \mathbb{C}(e_1 + \dots + e_n)$$

Notice that the matrix units E_{ij} are eigenvectors for $\operatorname{ad} h$, $h \in \mathfrak{h}$: $[h, E_{ij}] = (h_i - h_j) E_{ij}$

$$= (e_i - e_j)(h) E_{ij}.$$

Thus, the root decomposition is

$$R = \{e_i - e_j \mid i \neq j\} \subset \bigoplus \mathbb{C} e_i \Big/ \mathbb{C}(e_1 + \dots + e_n)$$

$$\mathfrak{g}_{e_i - e_j} = \mathbb{C} E_{ij}$$

The Killing form on $\mathfrak{h}$ is

$$(h, h') = \sum_{i \neq j} (h_i - h_j)(h_i' - h_j') = 2n \sum h_i h_i' = 2n \operatorname{tr}(hh')$$

Since the restriction of $(,)$ on $\mathfrak{h}$ is non-degenerate, we get an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ and a non-degenerate bilinear form on $\mathfrak{h}^*$, which we also denote by $(,)$. It can be expressed as follows

For $\alpha \in \mathfrak{h}^*$ denote by H_α the corresponding element of $\mathfrak{h}$

$$(\alpha, \beta) = \langle \beta, H_\alpha \rangle = (H_\alpha, H_\beta)$$

for $\alpha, \beta \in \mathfrak{h}^*$

Theorem Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$.

Let $(\, , \,)$ a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$.

1) R spans $\mathfrak{h}^*$ as a vector space

2) For each $\alpha \in R$, the root subspace $\mathfrak{g}_\alpha$ is one-dimensional.

3) For any two roots $\alpha, \beta \in R$

$$\langle \beta, h_\alpha \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$$

4) For $\alpha \in R$, the reflection operator $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$s_\alpha(\lambda) = \lambda - \langle \lambda, h_\alpha \rangle \alpha = \lambda - \frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \alpha.$$

Then for any roots α, β, $s_\alpha(\beta)$ is also a root.

5) For any root α, the only multiples of α which are also roots are $\pm\alpha$.

Representations of $\mathfrak{sl}(2,\mathbb{C})$

The idea is the same as before in the case of the adjoint representation:

We try to break up the representation into simultaneous eigenspaces of the elements of the Cartan subalgebra.

Theorem Every representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible.

Proof. $\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{su}(2,\mathbb{C}) \oplus i\,\mathfrak{su}(2,\mathbb{C})$, i.e.

$\mathfrak{sl}(2,\mathbb{C})$ is the complexification of $\mathfrak{su}(2,\mathbb{C})$.

$$\mathrm{Rep}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathrm{Rep}(\mathfrak{su}(2,\mathbb{C}))$$

$SU(2,\mathbb{C})$ is simply connected. So

$$\mathrm{Rep}(\mathfrak{su}(2,\mathbb{C})) \cong \mathrm{Rep}(SU(2,\mathbb{C}))$$

Weyl Every representation of a compact Lie group is completely reducible.

Idea. Use Haar measure to show that all representations are unitary.

Casimir element

$\mathfrak{g}$. $\qquad$ x_i basis of $\mathfrak{g}$

x^i dual basis

$$C = \sum x_i \otimes x^i$$

$$C \in Z(U\mathfrak{g})$$

$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ representation.

$\rho : U(\mathfrak{g}) \longrightarrow \mathfrak{gl}(V)$

C commutes with every $\rho(x)$ $x \in \mathfrak{g}$.

Definition Let V be a finite dim. representation of $\mathfrak{sl}(2, \mathbb{C})$. A vector $v \in V$ is called a vector of weight λ, $\lambda \in \mathbb{C}$, if it is an eigenvector for h with eigenvalue λ.

$$hv = \lambda v$$

We denote by $V[\lambda] \subset V$ the subspace of vectors λ.

Lemma

$$e \cdot V[\lambda] \subset V[\lambda+2]$$
$$f \, V[\lambda] \subset V[\lambda-2]$$

Proof. Let $v \in V[\lambda]$. Then

$$\begin{aligned} hev &= [h, e]v + eh \cdot v \\ &= 2ev + \lambda ev \\ &= (\lambda+2) ev \end{aligned}$$

So $ev \in V[\lambda+2]$. $\square$

Theorem Every finite dimensional representation V of $\mathfrak{sl}(2,\mathbb{C})$ can be written in the form

$$V = \bigoplus_{\lambda} V[\lambda]$$

Proof. Since every representation of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible, it suffices to prove this for irreducible V. Assume that V is irreducible.

Let $V' = \sum_{\lambda} V[\lambda]$. We know that eigenvectors with different eigenvalues are linearly independent. So $V' = \bigoplus V[\lambda]$. By the Lemma, V' is stable under e, f and h. Thus V' is a subrepresentation. Since V is irreducible, and $V' \neq 0$, $V' = V$ $\square$

Def. Let λ be a weight of V (i.e. $V[\lambda] \neq 0$) which is maximal:

$$\operatorname{Re} \lambda \geq \operatorname{Re} \lambda' \text{ for every weight } \lambda' \text{ of } V$$

Such a weight is called a <u>highest weight of V</u>. and vectors $v \in V[\lambda]$ will be called <u>highest weight vectors</u>.

<u>Lemma</u> Let $v \in V[\lambda]$ be a highest weight vector in V.

$$e.v \in V[\lambda+2]$$

1) $e.v = 0$

2) Let $v^k = \frac{f^k}{k!} v \quad k \geq 0$

Then

$$h v^k = (\lambda - 2k) v^k$$

$$f v^k = (k+1) v^{k+1}$$

$$e v^k = (\lambda - k + 1) v^{k-1}, \quad k > 0$$

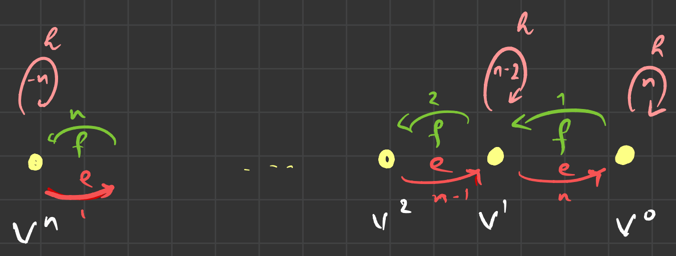
Theorem For any $n \geq 0$ let V_n be the finite dimensional vector space with basis $v^0, v^1, \dots, v^n$. Define the action of $sl(2, \mathbb{C})$ by

$$h v^k = (n-2k) v^k$$

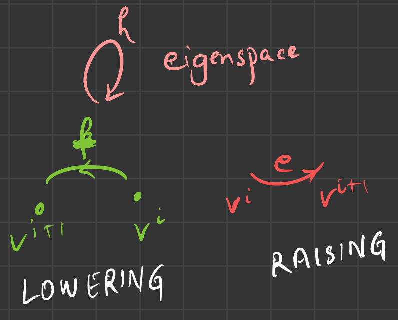
$$f v^k = (k+1) v^{k+1}, \quad k < n, \quad f v^n = 0$$

$$e v^k = (n+1-k) v^{k-1}, \quad k > 0, \quad e v^0 = 0$$

Then V_n is an irreducible rep. of $sl(2; \mathbb{C})$. For $n \neq m$ $V_n \not\cong V_m$. Every finite-dimensional irreducible representation of $sl(2, \mathbb{C})$ is isomorphic to some V_n.



IRREDUCIBLE REPRESENTATIONS OF $sl(2, \mathbb{C})$



Formulated differently, all the irreducible representations of $sl(2,\mathbb{C})$ are $\mathrm{Sym}^K(V)$ where V is the tautological two-dimensional representation of $sl(2,\mathbb{C})$.

$$\mathbb{C}[x,y]_h$$

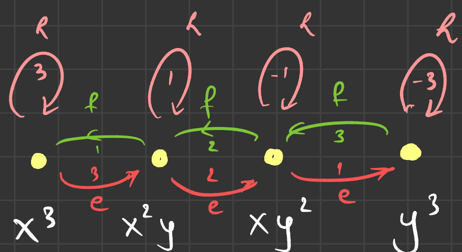
$$\mathrm{Sym}^d(V) \cong \mathbb{C}[x,y]_h^d$$

$$= \mathrm{span}\{x^d, x^{d-1}y, \ldots, xy^{d-1}, y^d\}$$

$$e.\ x^a y^b = a.\ x^{a-1}y^{b+1} = \left(y \cdot \frac{\partial(x^a y^b)}{\partial x}\right)$$

$$f.\ x^a y^b = b.\ x^{a+1}y^{b-1} = \left(x \cdot \frac{\partial}{\partial y} x^a y^b\right)$$

$$h.\ x^a y^b = (a-b)\, x^a y^b$$



REPRESENTATIONS OF LIE ALGEBRAS

Def. A <u>representation</u> of $\mathfrak{g}$ is a vector space V together with a morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$

Def A non-zero representation V of $\mathfrak{g}$ is called <u>irreducible</u> if it has no subrepresentations other than 0 or V.

Def. A representation is called <u>completely reducible</u> iff it is isomorphic to a direct sum of irreducible representations

ROOT SYSTEMS

Def. An abstract root system is a finite set of elements $R \subset E \setminus \{0\}$ where E is a Euclidean vector space (i.e. a real vector space with an inner product) s.t. the following properties hold.

(R1) R generates E as a vector space

(R2) For any two roots α, β the

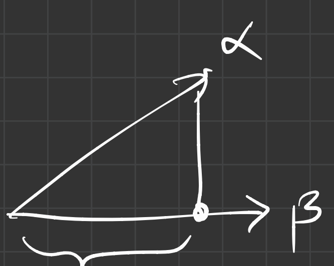
$$n_{\alpha\beta} = \frac{2(\alpha,\beta)}{(\beta,\beta)} \quad \in \mathbb{Z}$$

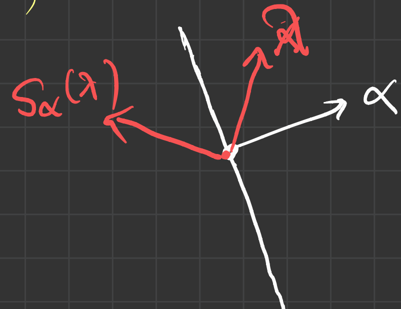
(R3) Let $s_\alpha : E \to E$ be defined by

$$s_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$$

Then for any roots α, β, $s_\alpha(\beta) \in R$

(R4) If α, $c\alpha$ are both roots, then $c = \pm 1$





Theorem Let $\mathfrak{g}$ be a semisimple complex Lie algebra with root decomposition. Then the set of roots $R \subset \mathfrak{h}^*_{\mathbb{R}} \setminus \{0\}$ is a reduced root system.

Def. For every root $\alpha \in R$, we define the coroot $\alpha^\vee \in E^*$ as

$$\langle \lambda, \alpha^\vee \rangle = \frac{2(\alpha, \lambda)}{(\alpha, \alpha)}$$

$$n_{\alpha\beta} = \langle \alpha, \beta^\vee \rangle$$

AUTOMORPHISMS AND WEYL GROUP

Def. Let $R_1 \subset E_1$, $R_2 \subset E_2$ be root systems. An isomorphism $\varphi: R_1 \longrightarrow R_2$ is a vector space isomorphism $\varphi: E_1 \longrightarrow E_2$ such that $\varphi(R_1) = R_2$ and $n_{\varphi(\alpha)\varphi(\beta)} = n_{\alpha\beta}$ for any $\alpha, \beta \in R_1$

Def. The <u>Weyl group W</u> of a root system R is the subgroup of $GL(E)$ generated by reflections s_α, $\alpha \in R$.