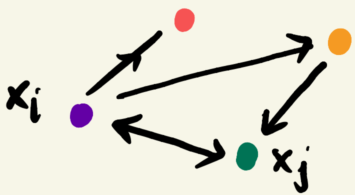
THRESHOLD LINEAR NETWORKS

SETUP



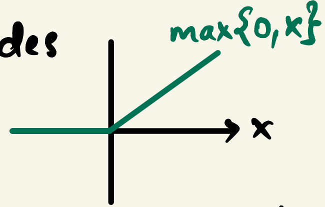
Network of neurons

- $x_i(t)$ represents the firing rate or activity level of node i.
- $$\frac{dx_i}{dt} = -x_i(t) + \left[\sum W_{ij} x_j(t) + b_i(t)\right]_+$$

or, $$\dot{\vec{x}} = -\vec{x} + \left[W\vec{x} + \vec{b}\right]_+$$

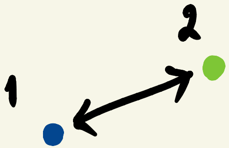
recurrent neural network

- W is a real valued matrix and it encodes the interaction strengths between nodes



- $[\cdot]_+ : \mathbb{R} \to \mathbb{R}$ is RELU function
- $b_i(t)$ is external input (usually this is constant)

EXAMPLE 0



Take $W = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$. Then

$$\boxed{\begin{aligned} \dot{x}_1 &= -x_1 + [-x_2 + \theta]_+ \\ \dot{x}_2 &= -x_2 + [2x_1 + \theta]_+ \end{aligned}}$$

($\theta = b$ is a positive real number)

INHIBITION DOMINATED TLNs

$$\frac{dx_i}{dt} = -x_i + \left[\sum W_{ij} x_j + b_j\right]_+ \quad (1)$$

RELU notation

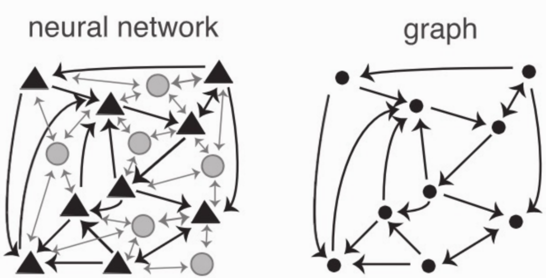
- take W to be entrywise nonpositive.
- $\Leftrightarrow$ interaction between neurons is inhibitory.

WHY?

[CURTO, MORRISON]

Figure 6.

CTLNs. A neural network with excitatory pyramidal neurons (triangles) and a background network of inhibitory interneurons (gray circles) that produces a global inhibition. The corresponding graph (right) retains only the excitatory neurons and their connections.



SPECIALIZATION (inhibition dominated TLNs)

COMBINATORIAL THRESHOLD LINEAR NETWORKS

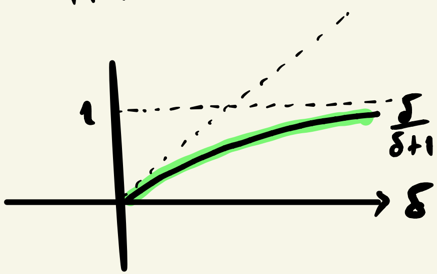
$$W_{ij} = \begin{cases} 0 & \text{if } i=j \\ -1+\epsilon & \text{if } j\to i \text{ in } G \\ -1-\delta & \text{if } j\not\to i \text{ in } G \end{cases}$$

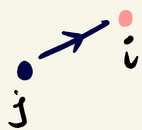
and $b_i = \theta$ positive constant

$$\begin{array}{c|c} & j \\ i & \begin{pmatrix} & \\ & -1+\epsilon \\ & \end{pmatrix} \end{array}$$

$0<\epsilon<1$

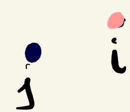
Here $\delta>0$, $0<\epsilon<\frac{\delta}{\delta+1}$.





Note:

- when $j\to i$ neuron j inhibits i less than itself.
- when $j\not\to i$ neuron j inhibits i more than it inhibits itself.

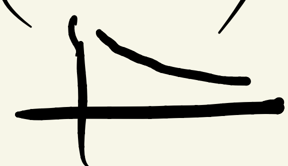


$$\begin{array}{c|c} & j \\ i & \begin{pmatrix} & \\ & -1-\delta \\ & \end{pmatrix} \end{array}$$

$$\frac{dx_i}{dt} = -x_i + [\ldots(-1-\delta)x_j]$$

$$\frac{dx_i}{dt} = -x_i + [\ldots\ldots(-1+\epsilon)x_j + \ldots]$$

$$\frac{dx_i}{dt} = -x_i$$



e^{-t}

MOTIVATING QUESTIONS

- Given the model defined by (1), what is the emergent dynamics?

fixed points? — stable? / unstable?

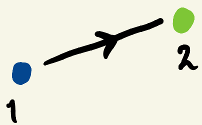
attracting sets? — periodic? / quasi-periodic? chaotic?

MEMORY

- What can the graph G tell about the dynamics?

$$(1) \qquad \dot{\vec{x}} = -\vec{x} + [W\vec{x} + \theta]_+$$

CTLNs AND HYPERPLANE ARRANGEMENTS



$$M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad W = \begin{pmatrix} 0 & -1-\delta \\ -1+\epsilon & 0 \end{pmatrix}$$

$\epsilon, \delta > 0 \qquad \theta > 0$

$0 < \epsilon < 1$

$$\dot{x}_1 = -x_1 + [(-1-\delta)x_2 + \theta]_+$$

$$\dot{x}_2 = -x_2 + [(-1+\epsilon)x_1 + \theta]_+$$

Set $(-1-\delta)x_2 + \theta = 0$

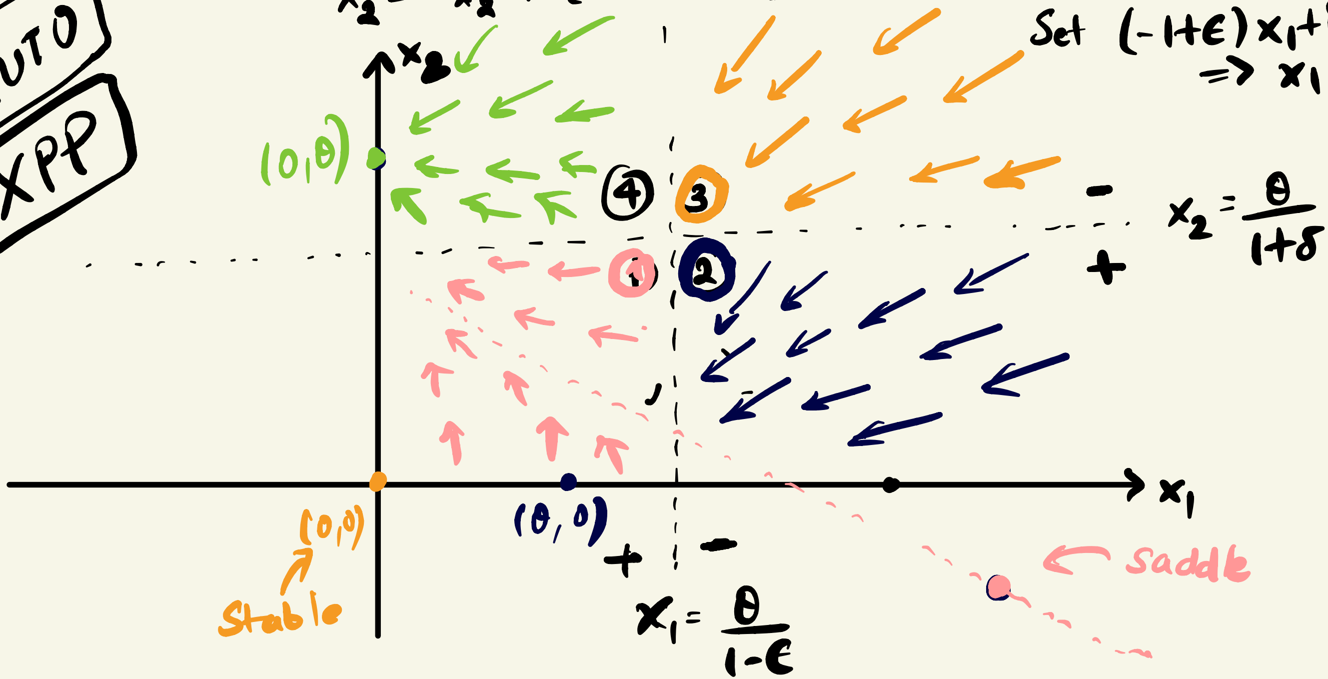
$\Rightarrow x_2 = \dfrac{\theta}{1+\delta}$

Set $(-1+\epsilon)x_1 + \theta = 0$

$\Rightarrow x_1 = \dfrac{\theta}{1-\epsilon}$

AUTO

XPP



1

$$\dot{x}_1 = -x_1 + (-1-\delta)x_2 + \theta$$
$$\dot{x}_2 = -x_2 + (-1+\epsilon)x_1 + \theta$$

fixed points $\Rightarrow \begin{cases} x_1 = (-1-\delta)x_2 + \theta \\ x_2 = (-1+\epsilon)x_1 + \theta \end{cases}$

$$\Rightarrow \begin{cases} x_1 = (-1-\delta)(-1+\epsilon)x_1 + (-1-\delta)\theta + \theta \\ \quad = (-1-\delta)(-1+\epsilon)x_1 - \cancel{\theta} - \delta\theta + \cancel{\theta} \\ \Rightarrow [(-1-\delta)(-1+\epsilon) - 1]x_1 = \delta\theta \\ \Rightarrow [\cancel{1} - \epsilon + \delta - \epsilon\delta - \cancel{1}]x_1 = \delta\theta \end{cases}$$

$$\Rightarrow x_1 = \frac{\delta\theta}{\delta - \epsilon - \epsilon\delta}$$

$$x_2 = \frac{(-1+\epsilon)\delta\theta}{\delta - \epsilon - \epsilon\delta} + \theta$$
$$= \frac{-\cancel{\delta\theta} + \epsilon\delta\theta + \cancel{\delta\theta} - \epsilon\theta - \cancel{\epsilon\theta\delta}}{\delta - \epsilon - \epsilon\delta}$$
$$= \frac{-\epsilon\theta}{\delta - \epsilon - \epsilon\delta}$$

$\left(\dfrac{\delta\theta}{\delta - \epsilon - \epsilon\delta}, \dfrac{-\epsilon\theta}{\delta - \epsilon - \epsilon\delta} \right)$ is a fixed point for

$$\begin{pmatrix} -1 & -1-\delta \\ -1+\epsilon & -1 \end{pmatrix}$$

$\Rightarrow$ saddle

$$\frac{-\epsilon\theta}{\delta-\epsilon-\epsilon\delta}$$ negative

$$\underbrace{\delta-\epsilon-\epsilon\delta}_{\text{denominator positive}}$$

$$\epsilon < \frac{\delta}{\delta+1}$$

$$\Rightarrow \epsilon\delta+\epsilon < \delta$$

$$\left.\begin{aligned}\dot{x}_1 &= -x_1 + (-1-\delta)x_2 + \theta \\ \dot{x}_2 &= -x_2\end{aligned}\right\}$$

fp: $\Rightarrow$ $x_2 = 0$

$x_1 = \theta$

$(\theta, 0)$ is a fixed point

$\Rightarrow$ stable

$$\begin{pmatrix} -1 & -1-\delta \\ 0 & -1 \end{pmatrix}$$

eigenvalues are -1 twice

$$\left.\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 \end{aligned}\right\}$$

f.p. $x_1 = 0$, $x_2 = 0$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow \text{stable}$$

4

$$\left.\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= -x_2 + (-1+\epsilon)x_1 + \theta \end{aligned}\right\}$$

f.p. $x_1 = 0$

$x_2 = \theta$

$(0, \theta)$ is a fixed point

$$\begin{pmatrix} -1 & 0 \\ -1+\epsilon & -1 \end{pmatrix} \Rightarrow \text{stable}$$


GO
TO
JUPYTER

SOME THEOR(Y/EMS)

THEOREM [Curto et al.]

Let G be a graph with no sinks. Then for any parameters $\varepsilon, \delta, \theta$ in the legal range, the associated CTLN has no stable fixed points.

THEOREM [MORRISON, DEGERATU, ITSKOV, CURTO]

Let (W, b) be a competitive nondegenerate TLN on n nodes, with $b_i > 0$ for all $i \in [n]$. Then

$$\sum_{\sigma \in FP(W,b)} idx(\sigma) = +1$$

$$idx(\sigma) = \text{sgn} \det (I - W_\sigma)$$

σ is support of fixed point
so some subset of $\{1, \dots, n\}$

In particular, the total number of fixed points $|FP(W,b)|$ is always odd.

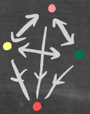
THEOREM [MDIC]

Let G be a simple directed graph, and consider an associated nondegenerate CTLN with $W = W(G, \varepsilon, \delta)$ for any choice of the parameters $\varepsilon, \delta, \theta > 0$ with $\varepsilon < 1$. If σ is a clique of G, then there exists a stable fixed point with support σ if and only if σ is target-free.

target free clique

$\Downarrow$

Stable fixed point with support



is a target

$\Downarrow$

there is no stable fixed point with support

REFERENCES

- Morrison, Degeratu, Itskov, Curto. Diversity of emergent dynamics in competitive threshold-linear networks. 2023
- Curto, Morrison. Graph rules for recurrent neural network dynamics. 2023

60's 70's

Hopfield network

+1 -1

$W = \begin{pmatrix} & \end{pmatrix}$

nonsymmetric

$\underline{W} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}$