# D-Modules

# DIFFERENTIAL OPERATORS

Let $X$ be a smooth algebraic variety over $\mathbb{C}$,
$\mathcal{O}_X$ the sheaf of regular functions called structure sheaf.
$\Theta_X$ be the sheaf of vector fields on $X$.

$$\Theta_X = \mathrm{Der}_{\mathbb{C}_X}(\mathcal{O}_X)$$
$$= \{\theta \in \mathrm{End}_{\mathbb{C}_X}(\mathcal{O}_X) \mid \theta(fg) = \theta(f)g + f\theta(g)\}$$

Since $X$ is smooth, the sheaf $\Theta_X$ is locally free of rank $n = \dim X$ over $\mathcal{O}_X$. We identify $\mathcal{O}_X$ with a subsheaf of $\mathrm{End}_{\mathbb{C}_X}(\mathcal{O}_X)$ by identifying $f \in \mathcal{O}_X$ with $g \longmapsto fg \in \mathcal{O}_X$

We define a sheaf $D_X$ as the $\mathbb{C}$-subalgebra of $\mathrm{End}_{\mathbb{C}_X}(\mathcal{O}_X)$ generate by $\mathcal{O}_X$ and $\Theta_X$.
This sheaf is called the <u>sheaf of differential operators</u> on $X$.

For any point of $X$ we can take its affine open neighborhood $U$ and a local coordinate system $\{x_i, \partial_i\}_{1 \le i \le n}$ on it satisfying

$$x_i \in \mathcal{O}_X(U), \quad \Theta_U = \bigoplus_{i=1}^{n} \mathcal{O}_U \partial_i$$

$$[\partial_i, \partial_j] = 0 \qquad [\partial_i, x_j] = \delta_{ij}$$

Hence,

$$D_U = D_X|_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathcal{O}_U \partial_x^\alpha$$

$$\left( \partial_x^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \right)$$

# D-Modules

Let $X$ be a smooth algebraic variety.

**Def.** A sheaf $M$ on $X$ is a left $D_X$-module if $M(U)$ is endowed with a left $D_X(U)$-module structure for each open subset $U$ of $X$ and these actions are compatible with restriction morphisms.

**Rmk:** $\mathcal{O}_X$ is a left $D_X$-module

**Lemma** Let $M$ be an $\mathcal{O}_X$-module. Giving a left $D_X$-module structure on $M$ extending the $\mathcal{O}_X$-module structure is equivalent to giving a $\mathbb{C}$-linear morphism

$$\nabla : \Theta_X \longrightarrow End_{\mathbb{C}}(M) \quad (\theta \mapsto \nabla_\theta)$$

satisfying

(i) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$ $\quad (f \in \mathcal{O}_X, \theta \in \Theta_X, s \in M)$

(ii) $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$

(iii) $\nabla_{[\theta_1, \theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$ $\quad (\theta_1, \theta_2 \in \Theta_X, s \in M)$

**Proof.** $[\theta, f] = \theta(f)$

(i) $\nabla_{f\theta}(s) = f\nabla_\theta(s)$ $\quad (f \in \mathcal{O}_X, \theta \in \Theta_X, s \in M)$

(ii) $\nabla_\theta(fs) = \theta(f)s + f\nabla_\theta(s)$

(iii) $\nabla_{[\theta_1,\theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s)$ $\quad (\theta_1, \theta_2 \in \Theta_X, s \in M)$

For a locally free left $\mathcal{O}_X$-module $M$ of finite rank, a $\mathbb{C}$-linear morphism $\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(M)$ satisfying (i) and (ii) is called a connection.

If it also satisfies (iii), it is called an integrable (or flat) connection.

**Def.** A $D_X$-module $M$ is an integrable connection if it is locally free of finite rank over $\mathcal{O}_X$.

**Prop.** There is a canonical equivalence

$$\tau : \mathcal{M}_\ell(X) \cong \mathcal{M}_r(X)$$

left $\mathcal{D}$-module $M$

$$\tau(M) = M \otimes_{\mathcal{O}(X)} \Omega(X)$$

$$R_v \big|_{\tau(M)} = -L_v \big|_M \otimes 1 \quad - 1 \otimes \mathrm{Lie}_v$$

# INVERSE AND DIRECT IMAGES

Let $\pi : X \longrightarrow Y$ be a morphism of smooth affine varieties.

This induces $\pi^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ making $\mathcal{O}(X)$ an $\mathcal{O}(Y)$ module.

Also, $\pi_* : TX \longrightarrow \pi^* TY$.

$$(x, \xi) \longmapsto (x, d\xi)$$

$$\begin{tikzcd} X \times TY \cong \pi^* TY \arrow[r] \arrow[d] & TY \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

This in turn induces a map on global sections

$$\pi_* : \mathrm{Vect}(X) \longrightarrow \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathrm{Vect}(Y)$$

Define

$$D_{X \to Y} = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} D(Y)$$

The left action of $\mathcal{O}(X)$ is the obvious one.

For $\theta \in \Theta_X = \mathrm{Vect}(X)$ let

$$\nabla_\theta (f \otimes s) = \theta(f) \otimes L + f \otimes \pi_*(\theta) s$$

$$\left( f \in \mathcal{O}(X),\ \in D(Y) \right)$$

The inverse image functor $\pi^\circ : M_\ell(Y) \longrightarrow M_\ell(X)$

is $$\pi^\circ(N) = D_{X\to Y} \otimes_{D(Y)} N$$

and direct image functor $\pi_0 : M_r(X) \longrightarrow M_r(Y)$ is

$$\pi_0(M) = M \otimes_{D(X)} D_{X\to Y}$$

These functors are right exact, so there are derived functors $L\pi_0$ and $L\pi^\circ$.

We denote $L\pi_0$ by $\pi_*$ and call it <u>full direct image</u>

Denote $L\pi^\circ[d] = \pi^!$ and call it <u>full inverse image</u>

$$d = \dim X - \dim Y$$

$X, Y$ irreducible.

# KASHIWARA'S THEOREM

Let $i : X \longrightarrow Y$ be a closed embedding of smooth varieties. In this case $D_{X\to Y} = D_Y / I_X D_Y$ where $I_X \subset \mathcal{O}_X$ is the ideal sheaf cutting out $Y$ inside $X$.

$\mathcal{O}(X) \otimes_{D(Y)} D(Y)$

View $X$ as a subvariety of $Y$ using $i$.

Locally $\{x_1, \dots, x_n, z_1, \dots, z_p\}$ s/

$$z_1 = \dots = z_p = 0$$

$$p = \dim Y - \dim X$$

$$i_0(M) = \bigoplus_{m_1, \dots, m_p \geq 0} M \partial_{z_1}^{m_1} \cdots \partial_{z_p}^{m_p}$$

$$M \otimes_{D(X)} D_{X\to Y}$$

$$M \otimes_{D(X)} D_Y / I_X D_Y$$

Given a closed subvariety $Z \subset Y$, we say $M \in \mathcal{M}(Y)$ is supported on $Z$ if for any $f \in \mathcal{O}(Y)$ vanishing on $Z$ and any local section $s$ of $M$, $\exists\, N \geq 0$ st $f^N s = 0$. Let $\mathcal{M}_Z(Y)$ denote the category of $\mathcal{D}$-modules on $Y$ supported on $Z$.

<u>Thm.</u> The functor $i_0 : \mathcal{M}(X) \longrightarrow \mathcal{M}_X(Y)$ is an equivalence of categories whose inverse is $i^!$.